Publication of this book was assisted
by a grant from the Publications Program
of the National Endowment for the Humanities,
an independent federal agency.

Chs: III, V, XVI

second edition

LOGIC, SEMANTICS, METAMATHEMATICS

PAPERS FROM 1923 TO 1938

BY

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Second edition edited and
introduced by JOHN CORCORAN

HACKETT PUBLISHING COMPANY

First edition published in 1956 by Oxford University Press

Printed in the United States of America
10 09 08 07 4 5 6 7 8

For further information, please address
Hackett Publishing Company, Inc.
P.O. Box 44937
Indianapolis, Indiana 46244-0937

Library of Congress Cataloging in Publication Data
Tarski, Alfred.
Logic, semantics, metamathematics.

Includes bibliographical references and indexes.
1. Logic, Symbolic and mathematical—Addresses, essays, lectures. 2. Semantics (Philosophy)—Address, essays, lectures. 1. Corcoran, John. II. Title.
BC135.T35 1983 160 83-10850
ISBN 0-915144-75-1

ISBN-13: 978-0-915144-75-4 (cloth)

The paper used in this publication meets the minimum requirements of American National Standard for Information Sciences—Permanence of Paper for Printed Library Materials, ANSI Z39.48-1984.
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TO THE MEMORY OF
HIS TEACHER
TADEUSZ KOTARBIŃSKI
THE AUTHOR

CONTENTS

Editor's Preface to the Revised Edition *vii*

Translator's Preface *ix*

Author's Acknowledgements *xiii*

Editor's Introduction to the Revised Edition *xv*

I. ON THE PRIMITIVE TERM OF LOGISTIC 1

II. FOUNDATIONS OF THE GEOMETRY OF SOLIDS 24

III. ON SOME FUNDAMENTAL CONCEPTS OF METAMATHEMATICS 30

IV. INVESTIGATIONS INTO THE SENTENTIAL CALCULUS 38
(by Jan ŁUKASIEWICZ and Alfred TARSKI)

V. FUNDAMENTAL CONCEPTS OF THE METHODOLOGY OF THE DEDUCTIVE SCIENCES 60

VI. ON DEFINABLE SETS OF REAL NUMBERS 110

VII. LOGICAL OPERATIONS AND PROJECTIVE SETS 143
(by Casimir KURATOWSKI and Alfred TARSKI)

VIII. THE CONCEPT OF TRUTH IN FORMALIZED LANGUAGES 152

Introduction 152

§ 1. The Concept of True Sentence in Everyday or Colloquial Language 154

§ 2. Formalized Languages, especially the Language of the Calculus of Classes 165

§ 3. The Concept of True Sentence in the Language of the Calculus of Classes 186

§ 4. The Concept of True Sentence in Languages of Finite Order 209

§ 5. The Concept of True Sentence in Languages of Infinite Order 241

§ 6. Summary 265

§ 7. Postscript 268

IX. SOME OBSERVATIONS ON THE CONCEPTS OF ω-CONSISTENCY AND ω-COMPLETENESS 279

X. SOME METHODOLOGICAL INVESTIGATIONS ON THE DEFINABILITY OF CONCEPTS 296

XI.	ON THE FOUNDATIONS OF BOOLEAN ALGEBRA	320
XII.	FOUNDATIONS OF THE CALCULUS OF SYSTEMS	342
XIII.	ON THE LIMITATIONS OF THE MEANS OF EXPRESSION OF DEDUCTIVE THEORIES (by Adolf LINDENBAUM and Alfred TARSKI)	384
XIV.	ON EXTENSIONS OF INCOMPLETE SYSTEMS OF THE SENTENTIAL CALCULUS	393
XV.	THE ESTABLISHMENT OF SCIENTIFIC SEMANTICS	401
XVI.	ON THE CONCEPT OF LOGICAL CONSEQUENCE	409
XVII.	SENTENTIAL CALCULUS AND TOPOLOGY	421
	ABBREVIATIONS	455
	BIBLIOGRAPHY	456
	ANALYTICAL INDEX	463
	INDEX OF NAMES OF PERSONS	503
	INDEX OF SYMBOLS	505

I

ON THE PRIMITIVE TERM OF LOGISTIC†

IN this article I propose to establish a theorem belonging to logistic concerning some connexions, not widely known, which exist between the terms of this discipline. My reasonings are based on certain sentences which are generally accepted among logisticians. But they do not depend on this or that particular theory of logical types. Among all the theories of types which could be constructed[1] there exist those according to which my arguments in their present form are perfectly legitimate.[2]

The problem of which I here offer a solution is the following: *is it possible to construct a system of logistic in which the sign of equivalence is the only primitive sign* (in addition of course to the quantifiers[3])?

This problem seems to me to be interesting for the following reason. We know that it is possible to construct the system of logistic by means of a single primitive term, employing for this purpose either the sign of implication, if we wish to follow the

[1] The possibility of constructing different theories of logical types is also recognized by the inventor of the best known of them. Cf. Whitehead, A. N., and Russell, B. (90), vol. 1, p. vii.

[2] Such a theory was developed in 1920 by S. Leśniewski in his course on the principles of arithmetic in the University of Warsaw; an exposition of the foundations of a system of logistic based upon this theory of types can be found in Leśniowski (46), (47), and (47 *b*).

[3] In the sense of Peirce (see Peirce, C. S. (58 *a*), p. 197) who gives this name to the symbols '$\prod$' (universal quantifier) and '$\sum$' (particular or existential quantifier) representing abbreviations of the expressions: 'for every signification of the terms . . .' and 'for some signification of the terms . . .'.

† BIBLIOGRAPHICAL NOTE. This article constitutes the essential part of the author's doctoral dissertation submitted to the University of Warsaw in 1923. The paper appeared in print in Polish under the title 'O wyrazie pierwotnym logistyki' in *Przegląd Filozoficzny*, vol. 26 (1923), pp. 68–89. A somewhat modified version was published in French in two parts under separate titles: 'Sur le terme primitif de la Logistique', *Fundamenta Mathematicae*, vol. 4 (1923), pp. 196–200, and 'Sur les *truth-functions* au sens de MM. Russell et Whitehead', ibid., vol. 5 (1924), pp. 59–74. The present English translation is based partly on the Polish and partly on the French original.

III

ON SOME FUNDAMENTAL CONCEPTS OF METAMATHEMATICS†

Our object in this communication is to define the meaning, and to establish the elementary properties, of some important concepts belonging to the *methodology of the deductive sciences*, which, following Hilbert, it is customary to call *metamathematics*.[1]

Formalized deductive disciplines form the field of research of metamathematics roughly in the same sense in which spatial entities form the field of research in geometry. These disciplines are regarded, from the standpoint of metamathematics, as sets of *sentences*. Those sentences which (following a suggestion of S. Leśniewski) are also called *meaningful sentences*, are themselves regarded as certain inscriptions of a well-defined form. The set of all sentences is here denoted by the symbol 'S'. From the sentences of any set X certain other sentences can be obtained by means of certain operations called *rules of inference*. These sentences are called the *consequences of the set X*. The set of all consequences is denoted by the symbol '$Cn(X)$'.‡

An exact definition of the two concepts, of sentence and of consequence, can be given only in those branches of metamathematics in which the field of investigation is a concrete formalized discipline. On account of the generality of the present considerations, however, these concepts will here be regarded as primitive and will be characterized by means of a series of

[1] Many ideas and results outlined in this report have been presented in a more detailed way in two later articles of the author, V and XII.

‡ More frequently they are now referred to as *sentences derivable from the set X*, while the term *consequences* is reserved for semantical consequences. See XVI (where the term 'logical consequence' is used instead of 'semantical consequence').

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† Bibliographical Note. The main ideas of this article were outlined by the author in a lecture to the Polish Mathematical Society, Warsaw Section, in 1928. For a summary of this lecture see Tarski, A. (72). The communication was presented (by J. Łukasiewicz) to the Warsaw Scientific Society on 27 March 1930; it was published under the title 'Über einige fundamentale Begriffe der Metamathematik' in *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, vol. 23, 1930, cl. iii, pp. 22–29.

axioms. In the customary notation of general set theory these axioms can be formulated in the following way:

Axiom 1. $\overline{\overline{S}} \leqslant \aleph_0$.

Axiom 2. *If $X \subseteq S$, then $X \subseteq Cn(X) \subseteq S$.*

Axiom 3. *If $X \subseteq S$, then $Cn(Cn(X)) = Cn(X)$.*

Axiom 4. *If $X \subseteq S$, then $Cn(X) = \sum_{Y \subseteq X \text{ and } \overline{\overline{Y}} < \aleph_0} Cn(Y)$.*

Axiom 5. *There exists a sentence $x \in S$ such that $Cn(\{x\}) = S$.*

With a view to reaching more profound results, other axioms of a more special nature are added to these. In contrast to the first group of axioms those of the second group apply, not to all deductive disciplines, but only to those which presuppose the sentential calculus, in the sense that in considerations relating to these disciplines we may use as premisses all true sentences of the sentential calculus.[1] In the axioms of this second group there occur as new primitive concepts two operations by means of which from simple sentences more complicated ones can be formed, namely the operations of forming implications and forming negations. The implication with the antecedent x and the consequent y is here denoted by the symbol '$c(x,y)$', and the negation of x by the symbol '$n(x)$'. The axioms are as follows:[2]

Axiom 6*. *If $x \in S$ and $y \in S$, then $c(x,y) \in S$ and $n(x) \in S$.*[3]

[1] i.e. all sentences which belong to the ordinary (two-valued) system of the sentential calculus; cf. IV, § 2, Def. 5.

[2] The numbering of the axioms of the second group and of the theorems which follow from them is distinguished from that of the remaining axioms and theorems by the presence of an asterisk '*'.

[3] As already explained, sentences are here regarded as material objects (inscriptions). From this standpoint the content of Ax. 6* does not correspond exactly to the intuitive properties of the concepts occurring in it. It is not always possible to form the implication of two sentences (they may occur in widely separated places). In order to simplify matters we have, in formulating this axiom, committed an error; this consists in *identifying equiform sentences* (as S. Leśniewski calls them). This error can be removed by interpreting S as the set of all types of sentences (and not of sentences) and by modifying in an analogous manner the intuitive sense of other primitive concepts. In this connexion by the type of a sentence x we understand the set of all sentences which are equiform with x.

Axiom 7*. *If* $X \subseteq S$, $y \in S$, $z \in S$ *and* $c(y,z) \in Cn(X)$, *then* $z \in Cn(X+\{y\})$.

Axiom 8*. *If* $X \subseteq S$, $y \in S$, $z \in S$ *and* $z \in Cn(X+\{y\})$, *then* $c(y,z) \in Cn(X)$.†

Axiom 9*. *If* $x \in S$, *then* $Cn(\{x, n(x)\}) = S$.

Axiom 10*. *If* $x \in S$, *then* $Cn(\{x\}) . Cn(\{n(x)\}) = Cn(0)$.

Axioms 8* and 10* are satisfied only when applied to those formalized disciplines in the sentences of which no free variables occur[1]. Instead of Ax. 7* and Ax. 8* in their full generality, it suffices to adopt as axioms the following particular cases of these statements:

Axiom 7′*. *If* $x \in S$, $y \in S$, $z \in S$, *and* $c(y,z) \in Cn(\{x\})$, then $z \in Cn(\{x,y\})$.

Axiom 8′*. *If* $x \in S$, $y \in S$, $z \in S$, *and* $z \in Cn(\{x,y\})$, *then* $c(y,z) \in Cn(\{x\})$.

In turn, Ax. 7′* can be equivalently replaced by a somewhat simpler statement:

Axiom 7″*. *If* $y \in S$ *and* $z \in S$ *then* $z \in Cn(\{y, c(y,z)\})$‡

On the basis of these axioms a series of theorems concerning the concepts involved can be proved, for example:

Theorem 1. *If* $X \subseteq Y \subseteq S$, *then* $Cn(X) \subseteq Cn(Y)$.

[1] This means that expressions (sentential functions) with free variables are not regarded as sentences.

† Ax. 8* is one of the formulations of what is known in the literature as the deduction theorem. This theorem, in its application to the formalism of *Principia Mathematica*, was first established by the author as far back as 1921 (in connexion with a discussion in the monograph of K. Ajdukiewicz (2)); the result was discussed in the author's lecture to the Warsaw Philosophical Institute, Section of Logic, listed by title in *Ruch Filozoficzny*, vol. 6 (1921–2), p. 72 *a*. Subsequently the deduction theorem was often applied in metamathematical discussion. Thus e.g. some theorems stated in the note (85) of A. Lindenbaum and A. Tarski, as well as in the note of A. Tarski (72), were obtained with the essential help of this result. In particular, Th. II in the first of these notes is simply a specialized form of the deduction theorem. A proof of the deduction theorem for a particular formalized theory is outlined in IX, p. 286 (proof of Th. 2 *a*). For reference to other appearances of the deduction theorem in the literature see a review by A. Church (of a book by W. V. Quine) in the *Journal of Symbolic Logic*, vol. 12 (1947), pp. 60–61.

‡ In the earlier editions of this paper it was claimed that Ax. 8′* can be equivalently replaced by a particular case of it, which is weaker than Ax. 8′*, in fact by the statement obtained from Ax. 8 by taking for X the empty set (and not a set with a single element). This claim, however, has proved to be erroneous; see Pogorzelski (60a), p. 168. Possible simplifications of Ax. 7 were not mentioned in earlier editions.

Theorem 2. *If* $X+Y \subseteq S$, *then*

$$Cn(X+Y) = Cn(X+Cn(Y)) = Cn(Cn(X)+Cn(Y)).$$

This theorem can be generalized to cover an arbitrary (even an infinite) number of summands.

Theorem 3*. *If* $x \in S$, $y \in S$ *and* $z \in S$, *then*

$$c(c(x,y), c(c(y,z), c(x,z))) \in Cn(0), \quad c(x, c(n(x), y)) \in Cn(0)$$

and

$$c(c(n(x),x),x) \in Cn(0).$$

This theorem asserts that every sentence obtained by substitution from one of the three axioms of the ordinary system of sentential calculus, which are due to J. Łukasiewicz,[1] is a consequence of the null set 0 (and hence is also a consequence of every set X of sentences). By the use of Ax. 7* this theorem can be extended to all substitution instances of tautologies, i.e., to all substitution instances of true sentences of the sentential calculus.

By means of the concepts S and $Cn(X)$ other important concepts of metamathematics can be defined. For example:

Definition 1. *A set X of sentences is called a* deductive system (*or simply a* system), *in symbols* $X \in \mathfrak{S}$, *if*

$$Cn(X) = X \subseteq S.$$

The following properties of systems are easily proved:

Theorem 4. *For every set* $X \subseteq S$ *there exists the smallest system* Y *which includes* X, *and in fact* $Y = Cn(X)$.

In consequence of this theorem the set $Cn(0)$ is the smallest system of all; this set can be called the *system of all logically true sentences*.

Theorem 5. *If* $\mathfrak{K} \subseteq \mathfrak{S}$ *and* $\mathfrak{K} \neq 0$, *then* $\prod_{X \in \mathfrak{K}} X \in \mathfrak{S}$ (*the intersection of any number of systems is itself a system*).

Theorem 6. *If for any two systems* $X \in \mathfrak{K}$ *and* $Y \in \mathfrak{K}$ *there is a system* $Z \in \mathfrak{K}$ *with* $X \subseteq Z$ *and* $Y \subseteq Z$, *and* $\mathfrak{K} \neq 0$, *then* $\sum_{X \in \mathfrak{K}} X \in \mathfrak{S}$.

Theorem 7* (of A. Lindenbaum). *If* $\mathfrak{K} \subseteq \mathfrak{S}$, $\overline{\overline{\mathfrak{K}}} < \aleph_0$ *and* $\sum_{X \in \mathfrak{K}} X \in \mathfrak{S}$, *then* $\sum_{X \in \mathfrak{K}} X \in \mathfrak{K}$; *in other words, no system can be represented as a sum of a finite number of systems distinct from itself.*

Theorem 8*. *If* $\mathfrak{K} \subseteq \mathfrak{S}$, $\overline{\overline{\mathfrak{K}}} < \aleph_0$, $Y \in \mathfrak{S}$, $Y \subseteq \sum_{X \in \mathfrak{K}} X$ *and* $Y \neq 0$, *then there is a system* $X \in \mathfrak{K}$ *such that* $Y \subseteq X$.

[1] Cf. IV, § 2.

We next introduce the notion of (*logical*) *equivalence*, together with the important notions of *consistency* and *completeness*.

DEFINITION 2. *The sets* X *and* Y *of sentences are called* (logically) equivalent, *in symbols* $X \sim Y$, *if* $X+Y \subseteq S$ *and*

$$Cn(X) = Cn(Y).$$

DEFINITION 3. *The set* X *of sentences is called* consistent, *in symbols* $X \in \mathfrak{W}$, *if* $X \subseteq S$ *and if the formula* $X \sim S$ *does not hold* (*i.e. if* $Cn(X) \neq S$).

DEFINITION 4. *The set* X *is said to be* complete, *in symbols* $X \in \mathfrak{V}$, *if* $X \subseteq S$ *and if every set* $Y \in \mathfrak{W}$ *which includes* X *satisfies the formula* $X \sim Y$.

With the help of the axioms of the second group it can be shown that Defs. 3 and 4 agree with the usual definitions of consistency and completeness:

THEOREM 9*. $X \in \mathfrak{W}$ *if and only if* $X \subseteq S$ *and if for no sentence* $y \in S$ *do we have both* $y \in Cn(X)$ *and* $n(y) \in Cn(X)$.

THEOREM 10*. $X \in \mathfrak{V}$ *if and only if* $X \subseteq S$ *and if for every sentence* $y \in S$ *at least one of the formulas* $y \in Cn(X)$ *and* $n(y) \in Cn(X)$ *holds.*

The following theorems are also derivable:

THEOREM 11. *If* $\mathfrak{K} \subseteq \mathfrak{W}$ *and if for every finite class* $\mathfrak{L} \subseteq \mathfrak{K}$ *a set* $Y \in \mathfrak{K}$ *exists which satisfies the formula* $\sum_{X \in \mathfrak{L}} X \subseteq Y$, *then* $\sum_{X \in \mathfrak{K}} X \in \mathfrak{W}$.

THEOREM 12 (of A. Lindenbaum). *If* $X \in \mathfrak{W}$, *then there is a set* $Y \in \mathfrak{S}.\mathfrak{W}.\mathfrak{V}$ *such that* $X \subseteq Y$; *in other words, every consistent set of sentences can be enlarged to form a consistent and complete system.*‡

THEOREM 13*. *Let* $X \subseteq S$. *In order that* $y \in Cn(X)$ *it is necessary and sufficient that* $y \in S$ *and that the formula* $X+\{n(y)\} \in \mathfrak{W}$ *should not hold.*

The concept of completeness is often confused with two other concepts which are related to it in content: that of *categoricity* and of *non-ramifiability*.[1] The next notion to be defined is closely related to that of completeness:

‡ This theorem was originally established by Lindenbaum for incomplete systems of sentential calculus. When his proof was subsequently reconstructed within a more general framework, in fact on the basis of the first group of our axioms, we have observed that, to assure the validity of the proof, it is necessary to insert in the axiom set a statement of a more special character than all the remaining axioms, viz. Ax. 5. Actually, a weaker statement to the effect that there is a finite set $X \subseteq S$ for which $Cn(X) = S$ would suffice for this purpose. It may be noticed that, on the basis of our full axiom set including both groups of axioms, Ax. 5 can be derived from other axioms and can therefore be omitted.

DEFINITION 5. *The* degree of completeness *of a set* $X \subseteq S$ of sentences, *in symbols* $\gamma(X)$, *is the smallest ordinal number* $\alpha \neq 0$ *which satisfies the following condition: there exists no increasing sequence of consistent non-equivalent sets* X_ξ *of sentences of type* α *which begins with* X (*i.e. no sequence of sets* X_ξ *satisfying the formulas:* $X_0 = X$, $X_\xi \subseteq X_\eta \subseteq S$ *and* $Cn(X_\xi) \neq Cn(X_\eta)$ *for* $\xi < \eta < \alpha$).

From this definition it follows that:

THEOREM 14. $\gamma(X) = 1$ *if and only if* $X \sim S$ (*i.e. if* $X \subseteq S$ *and if the formula* $X \in \mathfrak{W}$ *does not hold*); $\gamma(X) = 2$ *if and only if* $X \in \mathfrak{W}.\mathfrak{V}$; $\gamma(X) > 2$ *if and only if* $X \in \mathfrak{W}-\mathfrak{V}$.

Finally we introduce the following concepts:

DEFINITION 6. *A set* X *of sentences is called* independent, *in symbols* $X \in \mathfrak{U}$, *if* $X \subseteq S$ *and if* $Y = X$ *always follows from the formulas:* $Y \sim X$ *and* $Y \subseteq X$.

DEFINITION 7. *A set* Y *of sentences is called a* basis of the set X of sentences, *in symbols* $Y \in \mathfrak{B}(X)$, *if* $X \sim Y$ *and* $Y \in \mathfrak{U}$.

DEFINITION 8. *A set* Y *of sentences is called a* finite axiom system, *or for short an* axiom system, of the set X of sentences, *in symbols* $Y \in \mathfrak{Ax}(X)$, *if* $X \sim Y$ *and* $\overline{\overline{Y}} < \aleph_0$.

DEFINITION 9. *A set* X *of sentences is said to be* finitely axiomatizable, *or for short* axiomatizable, *in symbols* $X \in \mathfrak{A}$, *if* $\mathfrak{Ax}(X) \neq 0$.

THEOREM 15*. $X \in \mathfrak{U}$ *if and only if* $X \subseteq S$ *and for every* $y \in X$ *the formula* $X-\{y\}+\{n(y)\} \in \mathfrak{W}$ *holds.*

THEOREM 16. *If* $X \subseteq S$ *and* $\overline{\overline{X}} < \aleph_0$, *then there exists a set* $Y \subseteq X$ *such that* $Y \in \mathfrak{B}(X)$; *i.e. every finite set of sentences contains a basis as a subset.*

THEOREM 17*. *If* $X \subseteq S$, *then* $\mathfrak{B}(X) \neq 0$; *i.e. every set of sentences possesses a basis.*

[1] Cf. the remarks concerning these notions in article V, at the end of § 7.

THEOREM 18. *The following conditions are equivalent*: (1) $X \in \mathfrak{A}$; (2) *there is a set* $Y \subseteq X$, *such that* $Y \in \mathfrak{Ax}(X)$; (3) $\mathfrak{Ax}(X) \cdot \mathfrak{B}(X) \neq 0$; (4) *there exists a set* $Y \subseteq X$, *such that* $\overline{\overline{Y}} < \aleph_0$ *and* $Y \in \mathfrak{B}(X)$.

THEOREM 19*. *In order that* $X \in \mathfrak{A}$ *it is necessary and sufficient that* $X \subseteq S$ *and that X possess no infinite basis.*

THEOREM 20*. *Let* $X \in \mathfrak{S}$; *in order that* $X \in \mathfrak{A}$ *it is necessary and sufficient that no class* $\mathfrak{K} \subseteq \mathfrak{S}$ *exist which satisfies the following conditions*: $X \overline{\in} \mathfrak{K}$ *and* $X = \sum_{Y \in \mathfrak{K}} Y$ (*i.e. that X cannot be represented as a sum of systems distinct from itself*).

THEOREM 21. *We have* $\overline{\overline{\mathfrak{S}.\mathfrak{A}}} \leqslant \aleph_0$ *and* $\overline{\overline{\mathfrak{S}-\mathfrak{A}}} \leqslant \overline{\overline{\mathfrak{S}}} \leqslant 2^{\aleph_0}$; *if an infinite set* $X \in \mathfrak{U}$ *exists, then*

$$\overline{\overline{\mathfrak{S}.\mathfrak{A}}} = \aleph_0 \textit{ and } \overline{\overline{\mathfrak{S}-\mathfrak{A}}} = \overline{\overline{\mathfrak{S}}} = 2^{\aleph_0}.$$

It should be noted that in almost all known deductive disciplines an infinite and at the same time independent set of sentences can be constructed, which thus realizes the hypothesis of the second part of the last theorem. In these disciplines there are therefore more non-axiomatizable than axiomatizable systems; systems are, so to speak, only exceptionally axiomatizable.[1]

By some authors the concept of the independence of a set of sentences is sharpened in various directions (*complete independence* by E. H. Moore,[2] *maximal independence* by H. M. Sheffer[3]). These notions will not be discussed here.

Within the conceptual framework of this paper we can carry out metamathematical investigations relating to concrete deductive disciplines. For this purpose in each single case the concepts of sentence and of consequence must first be defined. We then take as a starting-point some set X of sentences in which we are

[1] Lindenbaum was the first to draw attention to this fact (in the domain of the sentential calculus, see IV, p. 51), thus stimulating interest in non-axiomatizable systems.
[2] "Introduction to a form of general analysis" in *The New Haven Mathematical Colloquium*. Yale University Press, New Haven 1910, p. 82.
[3] Sheffer, H. M. (63), p. 32.

interested. We investigate it from the point of view of consistency and axiomatizability; we try to determine its degree of completeness, and possibly to specify all the systems, in particular all the consistent and complete systems, which include X as a subset.†

† For an example of investigations in these directions which deal with the simplest deductive discipline, namely the sentential calculus, the reader is referred to IV. Some results concerning other deductive disciplines are briefly discussed in XII, § 5.

V

FUNDAMENTAL CONCEPTS OF THE METHODOLOGY OF THE DEDUCTIVE SCIENCES†

Introduction

THE deductive disciplines constitute the subject-matter of the *methodology of the deductive sciences*, which today, following Hilbert, is usually called *metamathematics*, in much the same sense in which spatial entities constitute the subject-matter of geometry and animals that of zoology. Naturally not all deductive disciplines are presented in a form suitable for objects of scientific investigation. Those, for example, are not suitable which do not rest on a definite logical basis, have no precise rules of inference, and the theorems of which are formulated in the usually ambiguous and inexact terms of colloquial language—in a word those which are not formalized. Metamathematical investigations are confined in consequence to the discussion of formalized deductive disciplines.

Strictly speaking metamathematics is not to be regarded as a single theory. For the purpose of investigating each deductive discipline a special metadiscipline should be constructed. The present studies, however, are of a more general character: their aim is *to make precise the meaning of a series of important metamathematical concepts* which are common to the special metadisciplines, *and to establish the fundamental properties of these concepts*. One result of this approach is that some concepts which can be defined on the basis of special metadisciplines will here be regarded as primitive concepts and characterized by a series of axioms.

An exact proof of the following results naturally requires, besides the above-mentioned axioms, a general logical basis.

† BIBLIOGRAPHICAL NOTE. This article originally appeared under the title 'Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. I', in *Monatshefte für Mathematik und Physik*, vol. 37 (1930), pp. 361–404. For earlier publications of the author on the same topics see the bibliographical note to III.

This basis should not be thought of as too comprehensive: for example, those chapters of the celebrated work of Whitehead and Russell[1] which comprise the sentential calculus, the theory of apparent variables, the calculus of classes, and the elements of the arithmetic of cardinal and ordinal numbers are quite sufficient for our purposes. The axiom of choice is not used in this discussion, and the axiom of infinity can also be easily eliminated.

As regards notation, we shall for practical reasons formulate these considerations in the terms of everyday language; but we shall also make use of a series of symbols, almost all of which are customary in textbooks and articles from the domain of set theory.

The variables 'x', 'y', 'z'... here denote individuals (objects of lowest type) and in particular sentences; the variables 'A', 'B', 'X', 'Y'..., sets of individuals; '$\mathfrak{K}$', '$\mathfrak{L}$'..., classes of sets (or systems of sets) and finally 'λ', 'μ', 'ξ'..., ordinal numbers. The symbols of the calculus of classes and arithmetic, '$\subseteq$,' '$+$', '$-$', '$\cdot$', '$\sum$', '$\prod$', '$<$', '$\leqslant$', '$\aleph_0$', etc., are used with their usual meanings; 'ω' denotes the smallest ordinal number of the second and 'Ω' of the third number class. The formulas '$x \in A$' or '$x \bar{\in} A$' express as usual that the set A respectively contains or does not contain the element x. '$\bar{\bar{A}}$' denotes the cardinal of the set A. The symbol '$\underset{f(x)}{E}[...]$' denotes the set of all values of the function f corresponding to those values of the argument x which satisfy the condition formulated in the brackets '[...]'. In particular we put $\{x\} = \underset{y}{E}[y = x]$ (the set which contains x as its only element), $\mathfrak{P}(A) = \underset{X}{E}[X \subseteq A]$ (the power set of A), $\mathfrak{Q}(A) = \underset{X}{E}[A \subseteq X]$, $\mathfrak{E} = \underset{X}{E}[\bar{\bar{X}} < \aleph_0]$ (the class of all finite, 'inductive' sets), and $\bar{\bar{\xi}} = \overline{\overline{\underset{\eta}{E}[\eta < \xi]}}$ (the cardinal number of the ordinal number ξ).

The results of the present article are applicable to arbitrary deductive disciplines, and in particular to the simplest deductive discipline, the sentential calculus. This article was originally

[1] Whitehead, A. N., and Russell, B. (90).

intended as the first part of a more comprehensive paper. The discussion in the second part dealt with those deductive disciplines which presuppose a certain logical basis, including at least the whole of the sentential calculus (approximately in the sense that all the valid statements of the sentential calculus can be used as premisses in all reasoning within these deductive disciplines). This second part was developed independently of the first part and was based upon a special axiom system. In its original form the second part has never appeared in print.[1]

In conclusion it should be noted that no particular philosophical standpoint regarding the foundations of mathematics is presupposed in the present work. Only incidentally, therefore I may mention that my personal attitude towards this question agrees in principle with that which has found emphatic expression in the writings of S. Leśniewski[2] and which I would call *intuitionistic formalism*.†

§ 1. Meaningful Sentences; Consequences of Sets of Sentences

From the standpoint of metamathematics every deductive discipline is a system of *sentences*, which we shall also call *meaningful sentences*.[3] The sentences are most conveniently regarded as inscriptions, and thus as concrete physical bodies. Naturally, not every inscription is a meaningful sentence of a given discipline: only inscriptions of a well-determined structure are regarded as meaningful. The concept of meaningful sentence has no fixed content, and must be relativized to a

[1] A modified version of the second part is to be found in XII. The discussion in Hertz, P. (27) has some points of contact with the present exposition. As examples of concrete investigations (within special metadisciplines) based upon the conceptual framework of this paper we may mention IV and Presburger, M. (61).

[2] Cf. especially Leśniewski, S. (46), especially p. 78.

[3] Instead of 'meaningful sentence' we could say 'well-formed sentence'. I use the word 'meaningful' to express my agreement with the doctrine of intuitionistic formalism mentioned above.

† This last sentence expresses the views of the author at the time when this article was originally published and does not adequately reflect his present attitude.

concrete formalized discipline. On the basis of the special metadisciplines mentioned in the introduction this concept can be reduced to intuitively simpler concepts;[1] in the present work, however, it must be taken as a primitive concept. The set of all meaningful sentences will be denoted by the symbol 'S'.

Let A be an arbitrary set of sentences of a particular discipline. With the help of certain operations, the so-called *rules of inference*, new sentences are derived from the set A, called the *consequences of the set* A. To establish these rules of inference, and with their help to define exactly the concept of consequence, is again a task of special metadisciplines; in the usual terminology of set theory the schema of such a definition can be formulated as follows:

The set of all *consequences of the set* A is the intersection of all sets which contain the set A and are closed under the given rules of inference.[2]

On account of the intended generality of these investigations we have no alternative but to regard the concept of consequence as primitive. *The two concepts—of sentence and of consequence—are the only primitive concepts which appear in this discussion.* The set of all consequences of the set A of sentences is here denoted by the symbol '$Cn(A)$'.

We will now state four axioms (Ax. 1–4) which express certain elementary properties of the primitive concepts and are satisfied in all known formalized disciplines.

We assume in the first place that the set S is at most denumerable.

Axiom 1. $\overline{\overline{S}} \leqslant \aleph_0$.

This axiom scarcely requires comment since the contrary hypothesis would be unnatural and would lead to undesired complications in proofs. Moreover, the question arises whether the assumption that the set S is infinite (even denumerably infinite) is consistent with the intuitive view of sentences as con-

[1] See IV, p. 39 (Def. 1) and p. 55 (Def. 8).

[2] Cf. IV, p. 40 (Def. 2) and p. 56 (Def. 9). A set is said to be *closed under given operations* if each of these operations, when applied to elements of the set, always yields again an element of the set (cf. Fréchet, M. (17)).

crete inscriptions. Without going deeper into this disputable question, which is irrelevant for our discussion, it may be noted here that I personally regard such an assumption as quite sensible, and that it appears to me even to be useful from a metamathematical standpoint to replace the inequality sign by the equality sign in Ax. 1.†

From the definition schema of the set of consequences formulated above it follows that every sentence which belongs to a given set is to be regarded as a consequence of that set, that the set of consequences of a set of sentences consists solely of sentences, and that the consequences of consequences are themselves consequences. These facts are expressed in the next two axioms.

AXIOM 2. *If* $A \subseteq S$, *then* $A \subseteq Cn(A) \subseteq S$.

AXIOM 3. *If* $A \subseteq S$, *then* $Cn(Cn(A)) = Cn(A)$.

Finally it should be noted that, in concrete disciplines, the rules of inference with the help of which the consequences of a set of sentences are formed are in practice always operations which can be carried out only on a finite number of sentences (usually even only on one or two sentences). Hence every consequence of a given set of sentences is a consequence of a finite subset of this set and vice versa. This can be shortly expressed by the following

AXIOM 4. *If* $A \subseteq S$, *then* $Cn(A) = \sum_{X \in \mathfrak{P}(A) \cdot \mathfrak{E}} Cn(X)$.

By way of example we give here some elementary consequences of the above axioms.

THEOREM 1. (*a*) *If* $A \subseteq B \subseteq S$, *then* $Cn(A) \subseteq Cn(B)$. [Ax. 4][1]

(*b*) *If* $A+B \subseteq S$, *then the formulas*

$$A \subseteq Cn(B) \quad \text{and} \quad Cn(A) \subseteq Cn(B)$$

are equivalent. [Ax. 2, Ax. 3, Th. 1*a*]

[1] In cases where a proof is clear I content myself with a reference to the theorems to be used.

† Compare the discussion of related problems in VIII, pp. 174 and 184.

In accordance with Th. 1*a* the operation Cn in the domain of sets of sentences is monotonic; this property can be expressed in several equivalent forms, e.g.:

$$\sum_{X \in \mathfrak{K}} Cn(X) \subseteq Cn\Big(\sum_{X \in \mathfrak{K}} X\Big), \quad \text{or} \quad Cn\Big(\prod_{X \in \mathfrak{K}} X\Big) \subseteq \prod_{X \in \mathfrak{K}} Cn(X),$$

for every non-null class $\mathfrak{K} \subseteq \mathfrak{P}(S)$.

THEOREM 2. (*a*) *If* $A+B \subseteq S$, *then*

$$Cn(A+B) = Cn(A+Cn(B)) = Cn(Cn(A)+Cn(B));$$

(*b*) *if, more generally,* $\mathfrak{K}$ *is an arbitrary class* $\subseteq \mathfrak{P}(S)$, *then*

$$Cn\Big(\sum_{X \in \mathfrak{K}} X\Big) = Cn\Big(\sum_{X \in \mathfrak{K}} Cn(X)\Big).$$

Proof. (*a*) According to Ax. 2,

$$A+B \subseteq A+Cn(B) \subseteq Cn(A)+Cn(B) \subseteq S,$$

whence by Th. 1*a*

(1) $Cn(A+B) \subseteq Cn(A+Cn(B)) \subseteq Cn(Cn(A)+Cn(B))$.

Again according to Th. 1*a* and Ax. 2 we have

$$Cn(A) \subseteq Cn(A+B) \subseteq S \quad \text{and} \quad Cn(B) \subseteq Cn(A+B) \subseteq S,$$

thus also $Cn(A)+Cn(B) \subseteq Cn(A+B) \subseteq S$ and therefore

(2) $Cn(Cn(A)+Cn(B)) \subseteq Cn(Cn(A+B))$.

Finally, from Ax. 3 it follows that

(3) $Cn(Cn(A+B)) = Cn(A+B)$.

The formulas (1) to (3) give immediately:

$$Cn(A+B) = Cn(A+Cn(B)) = Cn(Cn(A)+Cn(B)), \quad \text{q.e.d.}$$

(*b*) is proved in an analogous manner.

THEOREM 3. *If* $A \subseteq S$, $C \in \mathfrak{E}$, *and* $C \subseteq Cn(A)$, *then there exists a set* B *which satisfies the formulas* $B \in \mathfrak{E}$, $B \subseteq A$, *and* $C \subseteq Cn(B)$.

Proof. By the hypothesis and Ax. 4, a set $G(x)$ can be correlated uniquely with every element $x \in C$, in such a way that the formulas

(1) $x \in Cn(G(x))$, $G(x) \in \mathfrak{E}$, *and* $G(x) \subseteq A$ *for* $x \in C$

are satisfied. (Since the set C is finite, the existence of such a correlation can be established without using the axiom of choice).

It follows from (1) that

(2) $$C \subseteq \sum_{x \in C} Cn(G(x)).$$

We put

(3) $$B = \sum_{x \in C} G(x).$$

Since $C \in \mathfrak{E}$, we infer from (1) and (3) that

(4) $$B \in \mathfrak{E} \quad \textit{and} \quad B \subseteq A.$$

From (3) and (4) we also obtain $G(x) \subseteq B \subseteq A \subseteq S$ for $x \in C$. According to Th. 1*a* we thus have $Cn(G(x)) \subseteq Cn(B)$ for every element x of C, whence $\sum_{x \in C} Cn(G(x)) \subseteq Cn(B)$; and combining this inclusion with (2) we obtain

(5) $$C \subseteq Cn(B).$$

By (4) and (5) the set B satisfies the conclusion.

THEOREM 4. *Let $\mathfrak{K}$ be a class which satisfies the following condition: (α) for every finite subclass $\mathfrak{L}$ of $\mathfrak{K}$ there exists a set $Y \in \mathfrak{K}$, such that $\sum_{X \in \mathfrak{L}} X \subseteq Y$. If also $\mathfrak{K} \subseteq \mathfrak{P}(S)$, then*

$$Cn\Big(\sum_{X \in \mathfrak{K}} X\Big) = \sum_{X \in \mathfrak{K}} Cn(X).$$

Proof. Let x be a sentence such that

(1) $$x \in Cn\Big(\sum_{X \in \mathfrak{K}} X\Big).$$

Since by the hypothesis $\sum_{X \in \mathfrak{K}} X \subseteq S$, the existence of a set Z which satisfies the formulas

(2) $$Z \in \mathfrak{E}, \qquad Z \subseteq \sum_{X \in \mathfrak{K}} X,$$

and

(3) $$x \in Cn(Z)$$

follows from (1) and from Ax. 4. From (2) can further be inferred the existence of a finite subclass $\mathfrak{L}$ of $\mathfrak{K}$, such that $Z \subseteq \sum_{X \in \mathfrak{L}} X$. According to the premiss (α) there corresponds to the class $\mathfrak{L}$ a set $Y \in \mathfrak{K}$ such that $\sum_{X \in \mathfrak{L}} X \subseteq Y$. Consequently we have

(4) $$Y \in \mathfrak{K} \quad \textit{and} \quad Z \subseteq Y.$$

By Th. 1*a*, from (4) it follows that $Cn(Z) \subseteq Cn(Y)$, and therefore by (3) $x \in Cn(Y)$; hence with reference to (4) we obtain

(5) $$x \in \sum_{X \in \mathfrak{K}} Cn(X).$$

Thus we have shown that the formula (1) always implies (5), accordingly the following inclusion holds:

(6) $$Cn\Big(\sum_{X \in \mathfrak{K}} X\Big) \subseteq \sum_{X \in \mathfrak{K}} Cn(X).$$

On the other hand we have $Y \subseteq \sum_{X \in \mathfrak{K}} X \subseteq S$, whence according to Th. 1*a* $Cn(Y) \subseteq Cn\Big(\sum_{X \in \mathfrak{K}} X\Big)$ for every set $Y \in \mathfrak{K}$; consequently

(7) $$\sum_{X \in \mathfrak{K}} Cn(X) \subseteq Cn\Big(\sum_{X \in \mathfrak{K}} X\Big).$$

The formulas (6) and (7) give

$$Cn\Big(\sum_{X \in \mathfrak{K}} X\Big) = \sum_{X \in \mathfrak{K}} Cn(X), \quad \text{q.e.d.}$$

An immediate consequence of the last theorem is

COROLLARY 5. *Let $\mathfrak{K}$ be a class which satisfies the following condition: (α) $\mathfrak{K} \neq 0$ and for any two sets X and Y belonging to $\mathfrak{K}$ either $X \subseteq Y$ or $Y \subseteq X$. If $\mathfrak{K} \subseteq \mathfrak{P}(S)$, then*

$$Cn\Big(\sum_{X \in \mathfrak{K}} X\Big) = \sum_{X \in \mathfrak{K}} Cn(X). \qquad \text{[Th. 4]}$$

This corollary is often applied to the class of all terms of an increasing sequence of sets of sentences.

THEOREM 6. *Let $B + C \subseteq S$, and put*

$$F(X) = C \,.\, Cn(X+B)$$

for every set $X \subseteq S$ (and in particular $F(X) = Cn(X+B)$ in case $C = S$). We then have:

(*a*) $\overline{\overline{C}} \leqslant \aleph_0$;

(*b*) *if $A \subseteq C$, then $A \subseteq F(A) \subseteq C$;*

(*c*) *if $A \subseteq C$, then $F(F(A)) = F(A)$;*

(*d*) *if $A \subseteq C$, then $F(A) = \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} F(X)$.*

(In other words, Axs. 1–4 remain valid if 'S' and 'Cn' are everywhere replaced in them by 'C' and 'F' respectively.)

Proof. First let it be noted that in case $C = S$ we have, by Ax. 2, $Cn(X+B) \subseteq C$ for every $X \subseteq S$, so that the function $F(X) = C.Cn(X+B)$ reduces indeed to

$$F(X) = Cn(X+B).$$

(*a*) follows immediately from Ax. 1.

(*b*) is easily obtained from Ax. 2.

(*c*) By Ax. 2 and Th. 1*a* we have

$$C.Cn(A+B)+B \subseteq Cn(A+B)+B \subseteq S$$

and accordingly

$$Cn(C.Cn(A+B)+B) \subseteq Cn(Cn(A+B)+B);$$

in view of Th. 2*a*

$$Cn(Cn(A+B)+B) = Cn((A+B)+B) = Cn(A+B).$$

From this it follows that

$$C.Cn(C.Cn(A+B)+B) \subseteq C.Cn(A+B);$$

thus by the hypothesis

$$F(F(A)) \subseteq F(A).$$

Since the inverse inclusion $F(A) \subseteq F(F(A))$ results immediately from (*b*) (if we replace 'A' in it by '$F(A)$'), we finally obtain the desired formula $F(F(A)) = F(A)$.

(*d*) By Ax. 4:

(1) $$Cn(A+B) = \sum_{X \in \mathfrak{E}.\mathfrak{P}(A+B)} Cn(X).$$

For every set $X \in \mathfrak{P}(A+B)$ we obviously have $X \subseteq X_1+B$, where $X_1 = A.X \in \mathfrak{P}(A)$; from this by Th. 1*a* we have

$$Cn(X) \subseteq Cn(X_1+B).$$

If, moreover, $X \in \mathfrak{E}$, then $X_1 \in \mathfrak{E}$ and accordingly $X_1 \in \mathfrak{E}.\mathfrak{P}(A)$. Thus:

(2) $$\sum_{X \in \mathfrak{E}.\mathfrak{P}(A+B)} Cn(X) \subseteq \sum_{X \in \mathfrak{E}.\mathfrak{P}(A)} Cn(X+B).$$

On the other hand, if $X \in \mathfrak{P}(A)$, then $X+B \subseteq A+B$, whence $Cn(X+B) \subseteq Cn(A+B)$; therefore

(3) $$\sum_{X \in \mathfrak{E}.\mathfrak{P}(A)} Cn(X+B) \subseteq Cn(A+B).$$

The formulas (1)–(3) immediately give

$$Cn(A+B) = \sum_{X \in \mathfrak{E}.\mathfrak{P}(A)} Cn(X+B),$$

and thus

$$C.Cn(A+B) = \sum_{X \in \mathfrak{E}.\mathfrak{P}(A)} C.Cn(X+B).$$

By the hypothesis it follows from this that

$$F(A) = \sum_{X \in \mathfrak{E}.\mathfrak{P}(A)} F(X), \quad \text{q.e.d.}$$

In consequence of the above theorem, the concepts investigated in this discussion can be relativized in two distinct directions: 1. Instead of considering all meaningful sentences we can restrict our attention to the elements of a given set C of sentences—a *sentential domain*. 2. A fixed set of sentences B, a *basic set*, is chosen, and in the formation of the consequences of an arbitrary set X of sentences all the sentences of the basic set are added to the set X, so that the consequences of the set X in the new sense are identical with the consequences of the set $X+B$ in the old sense. According to Th. 6 (above) such a modification in the meaning of the primitive concepts does not affect the validity of the underlying axioms. Therefore all consequences of these axioms also remain valid, and in particular those theorems which will be presented in the following sections; one must only see to it that the non-primitive concepts in these theorems undergo an analogous relativization. On these grounds the theorem in question deserves the name *relativization theorem*.

§ 2. Deductive (Closed) Systems

With the help of the two concepts introduced in the preceding section, those of sentence and consequence, almost all basic concepts of metamathematics can be defined; on the basis of the given axiom system various fundamental properties of these concepts can be established.

In the first place an especially important category of sets of sentences will be singled out, namely the *deductive systems*. Every set of sentences which contains all its consequences is called a *deductive system*, or possibly a *closed system*, or simply a *system*.[1]

Deductive systems are, so to speak, organic units which form the subject matter of metamathematical investigations. Various important notions, like consistency, completeness, and axiomatizability, which we shall encounter in the sequel, are theoretically applicable to any sets of sentences, but in practice are applied chiefly to systems.

The class of all systems is denoted by the symbol '$\mathfrak{S}$'.

DEFINITION 1. $\mathfrak{S} = \underset{X}{E}[Cn(X) \subseteq X \subseteq S]$.

An easy transformation of the above definition yields

THEOREM 7. *In order that $A \in \mathfrak{S}$, it is necessary and sufficient that $Cn(A) = A \subseteq S$.* [Def. 1, Ax. 2]

Further properties of systems are expressed in the following theorems:

THEOREM 8. *If $B \in \mathfrak{S}$, then the conditions (α) $A \subseteq B$ and (β) $A \subseteq S$ and $Cn(A) \subseteq B$ are equivalent.* [Th. 1*b*, 7]

THEOREM 9. (*a*) *If $A \subseteq S$, then $Cn(A) \in \mathfrak{Q}(A).\mathfrak{S}$ and also $Cn(A) = \prod_{X \in \mathfrak{Q}(A).\mathfrak{S}} X$.*

(*b*) *In particular, $Cn(0) \in \mathfrak{S}$ and $Cn(0) = \prod_{X \in \mathfrak{S}} X$.*

Proof. (*a*) By Axs. 2 and 3,

$$A \subseteq Cn(A) \quad \text{and} \quad Cn(Cn(A)) = Cn(A) \subseteq S,$$

whence by Th. 7 $Cn(A) \in \mathfrak{Q}(A).\mathfrak{S}$. Accordingly

$$\prod_{X \in \mathfrak{Q}(A).\mathfrak{S}} X \subseteq Cn(A);$$

on the other hand from Th. 8 it follows that $Cn(A) \subseteq X$ for every set $X \in \mathfrak{Q}(A).\mathfrak{S}$, thus that $Cn(A) \subseteq \prod_{X \in \mathfrak{Q}(A).\mathfrak{S}} X$. Consequently we have $Cn(A) \in \mathfrak{Q}(A).\mathfrak{S}$ and $Cn(A) = \prod_{X \in \mathfrak{Q}(A).\mathfrak{S}} X$, q.e.d.

(*b*) results immediately from (*a*) if we put $A = 0$.

[1] The term 'deductive system' was used in earlier publications of the author; see III and the bibliographical note to that article. In Hertz, P. (27) 'closed system' is used; and in Zermelo, E. (92) we find the phrase 'logically closed system' used in the same sense.

In accordance with the above theorem $Cn(A)$ is the smallest closed system *over* A (i.e. the smallest of all systems which include the set A), and $Cn(0)$ is the smallest system in general. The system $Cn(0)$ can be called the *system of all logically provable* (or *logically valid*) *sentences*.

THEOREM 10. $S \in \mathfrak{S}$ *and* $S = \sum_{X \in \mathfrak{S}} X$. [Def. 1, Ax. 2]

S is thus the largest of all systems.

THEOREM 11. (*a*) *If $A \in \mathfrak{S}$ and $B \in \mathfrak{S}$, then $A.B \in \mathfrak{S}$.*

(*b*) *In general, if $\mathfrak{K} \subseteq \mathfrak{S}$ and $\mathfrak{K} \neq 0$, then $\prod_{X \in \mathfrak{K}} X \in \mathfrak{S}$.*

Proof. (*a*) From Def. 1 we obtain $Cn(A) \subseteq A \subseteq S$ and $Cn(B) \subseteq B \subseteq S$. By Th. 1*a* we have $Cn(A.B) \subseteq Cn(A)$ and $Cn(A.B) \subseteq Cn(B)$, thus $Cn(A.B) \subseteq Cn(A).Cn(B)$. Accordingly $Cn(A.B) \subseteq A.B \subseteq S$ whence, by Def. 1, $A.B \in \mathfrak{S}$, q.e.d.

(*b*) is proved analogously.

In contrast to the preceding theorem, a sum of systems is not always a new system. In this connexion only the following can be proved:

THEOREM 12. *If the class $\mathfrak{K}$ satisfies the condition (α) of Th. 4 and if $\mathfrak{K} \subseteq \mathfrak{S}$, then $\sum_{X \in \mathfrak{K}} X \in \mathfrak{S}$.*

Proof. By Def. 1 it follows from the formula $\mathfrak{K} \subseteq \mathfrak{S}$ that $\mathfrak{K} \subseteq \mathfrak{P}(S)$. The premisses of Th. 4 (or of Cor. 5) are thus satisfied, whence $Cn\left(\sum_{X \in \mathfrak{K}} X\right) = \sum_{X \in \mathfrak{K}} Cn(X)$.

From this last formula and the inclusion $\mathfrak{K} \subseteq \mathfrak{S}$ it follows by Th. 7 that $Cn\left(\sum_{X \in \mathfrak{K}} X\right) = \sum_{X \in \mathfrak{K}} X \subseteq S$, and consequently that $\sum_{X \in \mathfrak{K}} X \in \mathfrak{S}$, q.e.d.

The theorem above establishes a sufficient condition for a class of systems to be such that the sum of all systems of this class is itself a system. It can be shown that this condition is not necessary. However, if we restrict ourselves to those deductive disciplines which presuppose sentential calculus, and if we assume the class $\mathfrak{K}$ to be finite, then the converse of Th. 12 holds (a result of Lindenbaum).

§ 3. Logical (Inferential) Equivalence of Two Sets of Sentences

Two sets of sentences are called *logically* or *inferentially equivalent*, or simply *equivalent*, if they have all their consequences in common (i.e. their sets of consequences coincide). The significance of the concept consists in the fact that almost every property to be considered here applies to all sets equivalent to a given set A whenever it applies to A.

The class of all sets which are equivalent to a given set A of sentences is denoted by '$\mathfrak{Aq}(A)$'; the formula '$B \in \mathfrak{Aq}(A)$' accordingly states that the sets A and B are equivalent. But we shall not introduce a special relation symbol for the equivalence of sets.

Definition 2. $\mathfrak{Aq}(A) = \mathop{E}_{X}[A+X \subseteq S \text{ and } Cn(A) = Cn(X)]$.

Theorem 13. (*a*) *The formulas* $A \subseteq S$, $A \in \mathfrak{Aq}(A)$ *and* $\mathfrak{Aq}(A) \neq 0$ *are equivalent.* [Def. 2]

(*b*) *The formulas* $A \in \mathfrak{Aq}(B)$, $B \in \mathfrak{Aq}(A)$, $\mathfrak{Aq}(A) . \mathfrak{Aq}(B) \neq 0$ *and* $\mathfrak{Aq}(A) = \mathfrak{Aq}(B) \neq 0$ *are equivalent.* [Def. 2]

Theorem 14. *If* $A \subseteq B \subseteq C$ *and* $A \in \mathfrak{Aq}(C)$, *then* $B \in \mathfrak{Aq}(C)$ *and* $\mathfrak{Aq}(A) = \mathfrak{Aq}(B) = \mathfrak{Aq}(C)$.

Proof. By virtue of Def. 2 it follows from the hypothesis that $A \subseteq B \subseteq C \subseteq S$ and $Cn(A) = Cn(C)$; from this by Th. 1*a* we obtain $Cn(A) \subseteq Cn(B) \subseteq Cn(C)$ and $Cn(B) = Cn(C)$. By another application of Def. 2 we then obtain $B \in \mathfrak{Aq}(C)$, which finally on the basis of the hypothesis and of Th. 13*b* yields the second of the required formulas,

$$\mathfrak{Aq}(A) = \mathfrak{Aq}(B) = \mathfrak{Aq}(C).$$

Theorem 15. (*a*) *If* $A \in \mathfrak{Aq}(B)$ *and* $C \subseteq S$, *then*

$$A+C \in \mathfrak{Aq}(B+C).$$

(*b*) *If* $A_1 \in \mathfrak{Aq}(B_1)$ *and* $A_2 \in \mathfrak{Aq}(B_2)$, *then*

$$A_1+A_2 \in \mathfrak{Aq}(B_1+B_2).$$

(*c*) *In general, if to every set* $X \in \mathfrak{K}$ *there corresponds a set* $Y \in \mathfrak{L}$ *satisfying the formula* $X \in \mathfrak{Aq}(Y)$ *and vice versa, then*

$$\sum_{X \in \mathfrak{K}} X \in \mathfrak{Aq}\Big(\sum_{Y \in \mathfrak{L}} Y\Big).$$

Proof. (*a*) By means of Def. 2 the hypothesis yields

$$(A+C)+(B+C) \subseteq S \quad \text{and} \quad Cn(A) = Cn(B).$$

From this by applying Th. 2*a* we obtain

$$Cn(A+C) = Cn(Cn(A)+C) = Cn(Cn(B)+C) = Cn(B+C).$$

Accordingly by Def. 2 we have $A+C \in \mathfrak{Aq}(B+C)$, q.e.d.

(*b*) and (*c*) follow in an analogous way by means of Th. 2*a*, *b*.

Theorem 16. *For every set* $A \subseteq S$ *there is a corresponding sequence of sentences* x_ν *of type* $\pi \leqslant \omega$[1] *which satisfies the formulas*

$$(\alpha) \quad x_\nu \,\overline{\in}\, Cn\Big(\mathop{E}_{x_\mu}[\mu < \nu]\Big) \text{ for every } \nu < \pi$$

and

$$(\beta) \quad \mathop{E}_{x_\nu}[\nu < \pi] \in \mathfrak{Aq}(A).$$

Proof. In consequence of Axs. 1 and 2 as well as of the hypothesis of the theorem, the set $Cn(A) \subseteq S$ is at most denumerable; the elements of this set can therefore be ordered in an infinite sequence (with possibly repeating terms) of the type ω, so that

(1) $$Cn(A) = \mathop{E}_{y_\lambda}[\lambda < \omega] \subseteq S.$$

Let

(2) *λ_0 be the smallest of the numbers $\lambda < \omega$ which satisfy the formula*

$$y_\lambda \,\overline{\in}\, Cn(0);$$

Further, let

(3) *for $0 < \nu < \omega$, λ_ν be the smallest of the numbers $\lambda < \omega$ which satisfy the formula*

$$y_\lambda \,\overline{\in}\, Cn\Big(\mathop{E}_{y_{\lambda_\mu}}[\mu < \nu]\Big);$$

Finally, let

(4) *π be the smallest ordinal number with which the conditions (2) and (3) do not correlate any number λ_π.*

We put

(5) *$x_\nu = y_{\lambda_\nu}$ for $\nu < \pi$.*

[1] Where possibly $\pi = 0$ (the empty sequence, which has no terms at all, is regarded as a sequence of type 0).

It follows from (2)–(4) that

(6) $$\pi \leqslant \omega.$$

From (1)–(5) we easily obtain

(7) $$x_\nu \overline{\in} Cn\Big(\underset{x_\mu}{E}[\mu < \nu]\Big) \quad \textit{for } \nu < \pi$$

and

(8) $$\underset{x_\nu}{E}[\nu < \pi] \subseteq Cn(A).$$

Let us suppose that

(9) *there exist numbers* $\lambda < \omega$ *such that* $y_\lambda \overline{\in} Cn\Big(\underset{x_\nu}{E}[\nu < \pi]\Big)$.

Accordingly (in view of (5)) let

(10) λ' *bé the smallest of the numbers* λ *which satisfy the formulas* $\lambda < \omega$ *and* $y_\lambda \overline{\in} Cn\Big(\underset{y_{\lambda_\nu}}{E}[\nu < \pi]\Big)$.

If π were less than ω, then, by comparing the statements (2), (3), and (10), we should obtain $\lambda' = \lambda_\pi$, which contradicts the condition (4). Consequently, on the basis of (6),

(11) $$\pi = \omega.$$

In accordance with Th. 1 *a* and with reference to (1)–(3) we have $Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu]\Big) \subseteq Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \pi]\Big)$ for every $\nu < \pi$. We thus have by (10) $y_{\lambda'} \overline{\in} Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu]\Big)$ for $\nu < \pi$ and in particular $y_{\lambda'} \overline{\in} Cn(0)$ for $\nu = 0$, whence, in view of (2) and (3),

(12) $$\lambda' \geqslant \lambda_\nu \quad \text{for every } \nu < \pi.$$

Finally it is easy to show that

(13) $$\lambda_{\nu_1} < \lambda_{\nu_2} \quad \textit{for } \nu_1 < \nu_2 < \pi.$$

In fact, let $\nu_1 < \nu_2 < \pi$. By Ax. 2 and Th. 1 *a*, the conditions (1)–(4) imply then the following formulas:

$$\underset{y_{\lambda_\mu}}{E}[\mu < \nu_2] \subseteq Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu_2]\Big)$$

and $$Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu_1]\Big) \subseteq Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu_2]\Big).$$

A consequence of the first inclusion is

$$y_{\lambda_{\nu_1}} \in Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu_2]\Big);$$

since by (3) (for $\nu = \nu_2$) $y_{\lambda_{\nu_2}} \overline{\in} Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu_2]\Big)$, the numbers λ_{ν_1} and λ_{ν_2} cannot be identical. By combining the last formula with the second inclusion we obtain

$$y_{\lambda_{\nu_2}} \overline{\in} Cn\Big(\underset{y_{\lambda_\mu}}{E}[\mu < \nu_1]\Big).$$

In other words λ_{ν_2} is one of the numbers λ which satisfy the condition (3) for $\nu = \nu_1$ (or the condition (2) in the case $\nu_1 = 0$); since λ_{ν_1} is by definition the smallest of these numbers, we have $\lambda_{\nu_1} \leqslant \lambda_{\nu_2}$. We thus finally reach the desired formula $\lambda_{\nu_1} < \lambda_{\nu_2}$.

From the formulas (11)–(13) we conclude at once that $\lambda' \geqslant \omega$, which contradicts (10) and hence refutes the assumption (9). It must thus be accepted that

(14) $$y_\lambda \in Cn\Big(\underset{x_\nu}{E}[\nu < \pi]\Big) \quad \textit{for every } \lambda < \omega.$$

From (1) and (14) it follows that

$$Cn(A) \subseteq Cn\Big(\underset{x_\nu}{E}[\nu < \pi]\Big);$$

the application of Th. 1 *b* to the inclusion (8) gives

$$Cn\Big(\underset{x_\nu}{E}[\nu < \pi]\Big) \subseteq Cn(A).$$

Thus we have the identity

$$Cn(A) = Cn\Big(\underset{x_\nu}{E}[\nu < \pi]\Big),$$

which in accordance with Def. 2 leads to the formula

(15) $$\underset{x_\nu}{E}[\nu < \pi] \in \mathfrak{Aq}(A).$$

The formulas (6), (7), and (15) state that the sequence of sentences x_ν satisfies all conditions of the conclusion.

In the proof above we have used a method of reasoning which is repeatedly employed in set-theoretical considerations. For the idea of applying it to metamathematical problems we are indebted to Lindenbaum, who employed it in the proof of Th. 56 given below.

A sequence of sentences which satisfies the condition (α) of Th. 16 could be called *ordinally independent*; if at the same time the formula (β) is satisfied, then this sequence is to be called an *ordered basis* of the set A of sentences. In this terminology the theorem states that every set of sentences possesses an ordered basis.

THEOREM 17. *If* $A \subseteq S$, *then* $Cn(A) \in \mathfrak{Aq}(A)$ *and*

$$Cn(A) = \sum_{X \in \mathfrak{Aq}(A)} X. \quad \text{[Def. 2, Axs. 2, 3]}$$

THEOREM 18. (*a*) *In order that* $A \in \mathfrak{Aq}(B)$ *and* $B \in \mathfrak{S}$, *it is necessary and sufficient that* $A \subseteq S$ *and* $B = Cn(A)$.

[Def. 2, Ths. 7, 9 *a*]

(*b*) *If* $A \subseteq S$, *then* $\{Cn(A)\} = \mathfrak{S}.\mathfrak{Aq}(A)$; [Ths. 13 *a*, 18 *a*]

(*c*) *If* $A \in \mathfrak{S}$, *then* $\{A\} = \mathfrak{S}.\mathfrak{Aq}(A)$. [Ths. 7, 18 *a*, *b*]

By Ths. 17 and 18 *a*, $Cn(A)$ is both the greatest of all sets equivalent to the set A and the only system equivalent to A; consequently the set $Cn(A)$ can be regarded and used as a representative of the whole class $\mathfrak{Aq}(A)$.

§ 4. AN AXIOM SYSTEM OF A SET OF SENTENCES, AXIOMATIZABLE SETS OF SENTENCES

An *axiom system* of a set of sentences is a finite set which is equivalent to that set; a set of sentences which possesses at least one axiom system is called *axiomatizable*. Lindenbaum has directed our attention to the concept of axiomatizability. In his investigations on the meta-sentential-calculus he has established the interesting fact that, in addition to axiomatizable sets of sentences, non-axiomatizable sets also exist.[1]

Using '$\mathfrak{Ax}(A)$' to denote the class of all axiom systems of a set A of sentences, and '$\mathfrak{A}$' to denote the class of all axiomatizable sets of sentences, we reach

DEFINITION 3. (*a*) $\mathfrak{Ax}(A) = \mathfrak{E}.\mathfrak{Aq}(A)$.

(*b*) $$\mathfrak{A} = \underset{X}{E}[\mathfrak{Ax}(X) \neq 0].$$

The following consequences of this definition are easily provable:

THEOREM 19. (*a*) *In order that* $A \in \mathfrak{Ax}(A)$ *it is necessary and sufficient that* $A \in \mathfrak{E}.\mathfrak{P}(S)$. [Def. 3 *a*, Th. 13 *a*]

(*b*) $\mathfrak{E}.\mathfrak{P}(S) \subseteq \mathfrak{A}$. [Th. 19 *a*, Def. 3 *b*]

[1] Cf. IV, p. 51 (Th. 27).

THEOREM 20. (*a*) *If* $\mathfrak{Ax}(A).\mathfrak{Ax}(B) \neq 0$, *then* $A \in \mathfrak{Aq}(B)$.

[Def. 3 *a*, Th. 13 *b*]

(*b*) *If* $A \in \mathfrak{Aq}(B)$, *then* $\mathfrak{Ax}(A) = \mathfrak{Ax}(B)$. [Th. 13 *b*, Def. 3 *a*]

(*c*) *If* $A \in \mathfrak{A}$ *then* $\mathfrak{Aq}(A) \subseteq \mathfrak{A}$.

THEOREM 21. (*a*) *If* $A_1 \in \mathfrak{Ax}(B_1)$ *and* $A_2 \in \mathfrak{Ax}(B_2)$, *then* $A_1 + A_2 \in \mathfrak{Ax}(B_1 + B_2)$. [Def. 3 *a*, Th. 15 *b*]

(*b*) *If* $A \in \mathfrak{A}$ *and* $B \in \mathfrak{A}$, *then* $A + B \in \mathfrak{A}$.

[Def. 3 *b*, Th. 21 *a*]

THEOREM 22. *If* $A \in \mathfrak{A}$, *then* $\mathfrak{P}(A).\mathfrak{Ax}(A) \neq 0$.

Proof. By Defs. 2 and 3 it follows from the hypothesis that there is a set X which satisfies the formulas

(1) $$X \in \mathfrak{E},$$

(2) $$A + X \subseteq S \quad \text{and} \quad Cn(A) = Cn(X).$$

From (2) with the help of Ax. 2 we obtain $A \subseteq S$ and $X \subseteq Cn(A)$; from this, by applying Th. 3 and by reference to (1), we infer that there exists a set Y such that

(3) $$Y \in \mathfrak{E},$$

(4) $$Y \subseteq A \subseteq S \quad \text{and} \quad X \subseteq Cn(Y).$$

By Th. 1 *a*, *b* the formulas (4) imply

$$Cn(X) \subseteq Cn(Y) \subseteq Cn(A),$$

whence, in view of (2), $Cn(A) = Cn(Y)$; and from this by means of Def. 2 we obtain:

(5) $$Y \in \mathfrak{Aq}(A).$$

By Def. 3 *a* the formula $Y \in \mathfrak{Ax}(A)$ follows from (3) and (5). By combining this formula with (4) we obtain at once

$$\mathfrak{P}(A).\mathfrak{Ax}(A) \neq 0, \quad \text{q.e.d.}$$

THEOREM 23. *If the class* $\mathfrak{K}$ *satisfies the condition* (α) *of Theorem* 4 (*or of Cor.* 5) *and if* $\sum_{X \in \mathfrak{K}} X \in \mathfrak{A}$, *then*

$$\mathfrak{K}.\mathfrak{Aq}\Big(\sum_{X \in \mathfrak{K}} X\Big) \neq 0.$$

Proof. By Th. 22 it follows from the hypothesis that

$$\mathfrak{P}\Big(\sum_{X \in \mathfrak{K}} X\Big).\mathfrak{Ax}\Big(\sum_{X \in \mathfrak{K}} X\Big) \neq 0.$$

Hence by Def. 3 *a* we have, for some Y,

(1) $$Y \subseteq \sum_{X \in \mathfrak{K}} X,$$

(2) $$Y \in \mathfrak{E},$$

and

(3) $$Y \in \mathfrak{Aq}\Big(\sum_{X \in \mathfrak{K}} X\Big).$$

From (1) and (2) it is easily inferred that there is a class $\mathfrak{L} \in \mathfrak{E}.\mathfrak{P}(\mathfrak{K})$ which satisfies the formula $Y \subseteq \sum_{X \in \mathfrak{L}} X$. Since by hypothesis the condition (α) of Th. 4 (or of Cor. 5) is satisfied, it follows that there is a set $Z \in \mathfrak{K}$, which includes the set $\sum_{X \in \mathfrak{L}} X$ and *a fortiori* the set Y. We thus have

(4) $$Z \in \mathfrak{K},$$

(5) $$Y \subseteq Z \subseteq \sum_{X \in \mathfrak{K}} X.$$

By Th. 14 the formulas (3) and (5) give at once

(6) $$Z \in \mathfrak{Aq}\Big(\sum_{X \in \mathfrak{K}} X\Big).$$

Finally from (4) and (6) we obtain

$$\mathfrak{K}.\mathfrak{Aq}\Big(\sum_{X \in \mathfrak{K}} X\Big) \neq 0, \quad \text{q.e.d.}$$

THEOREM 24. *Let a sequence of sentences x_ν of the type $\pi \leqslant \omega$ be given which satisfies the formula* (α) $x_\nu \in S - Cn\Big(\underset{x_\mu}{E}[\mu < \nu]\Big)$ *for every $\nu < \pi$. In order that $\underset{x_\nu}{E}[\nu < \pi] \in \mathfrak{A}$, it is necessary and sufficient that $\pi < \omega$.*

Proof. A. Let us suppose that

(1) $$\underset{x_\nu}{E}[\nu < \pi] \in \mathfrak{A},$$

and nevertheless assume

(2) $$\pi = \omega.$$

By Th. 22 and Def. 3 *a* it follows from (1) that there is a set Y which satisfies the formulas

(3) $$Y \subseteq \underset{x_\nu}{E}[\nu < \pi], \qquad Y \in \mathfrak{E},$$

and

(4) $$Y \in \mathfrak{Aq}\Big(\underset{x_\nu}{E}[\nu < \pi]\Big).$$

Moreover, from (3) we infer the existence of a number ν such that

(5) $$\nu < \omega \quad \textit{and} \quad Y \subseteq \underset{x_\mu}{E}[\mu \leqslant \nu].$$

The formulas (2) and (5) give

(6) $$\nu + 1 < \pi,$$

whence by the hypothesis of the theorem we have

(7) $$x_{\nu+1} \mathrel{\overline{\in}} Cn\Big(\underset{x_\mu}{E}[\mu \leqslant \nu]\Big).$$

Since $\underset{x_\mu}{E}[\mu \leqslant \nu] \subseteq S$, we obtain from (5) with the help of Th. 1 *a* $Cn(Y) \subseteq Cn\Big(\underset{x_\mu}{E}[\mu \leqslant \nu]\Big)$; by comparing this inclusion with (7) we reach the formula

(8) $$x_{\nu+1} \mathrel{\overline{\in}} Cn(Y).$$

By Ax. 2 and Def. 3 it results from (4) that

$$\underset{x_\nu}{E}[\nu < \pi] \subseteq Cn\Big(\underset{x_\nu}{E}[\nu < \pi]\Big) = Cn(Y).$$

Thus by virtue of (6) we have $x_{\nu+1} \in Cn(Y)$, which obviously contradicts the formula (8) and refutes the assumption (2).

Hence we conclude that

(9) $$\pi < \omega.$$

B. If on the other hand the inequality (9) is satisfied, then according to the hypothesis we have $\underset{x_\nu}{E}[\nu < \pi] \in \mathfrak{E}.\mathfrak{P}(S)$, from which by Th. 19 *b* the formula (1) follows.

We have thus established the equivalence of the formulas (1) and (9), and this concludes the proof.

THEOREM 25. *The following conditions are equivalent:* (α) $A \in \mathfrak{A}$; (β) *there is no sequence of sentences x_ν of type ω which satisfies both of the formulas* (α) *and* (β) *of Th.* 16, *and in addition* $A \subseteq S$; (γ) *there is a sequence of sentences x_ν of the type $\pi < \omega$, which satisfies both of these formulas.*

Proof. If the condition (α) holds, then by Th. 20c we have $\mathfrak{Aq}(A) \subseteq \mathfrak{A}$ and accordingly $\underset{x_\nu}{E}[\nu < \pi] \in \mathfrak{A}$ for every sequence of sentences which satisfies the conclusion of Th. 16; from Th. 24 it thus results that the inequality $\pi \geqslant \omega$ does not hold. But since $A \subseteq S$ (by Defs. 2 and 3a, b), we can assert that (β) follows from (α). On the other hand by Th. 16 there exists for every set $A \subseteq S$ a sequence of sentences either of type ω or of type $\pi < \omega$, which satisfies the conclusion of this theorem; (β) thus implies (γ). Finally, if (γ) is satisfied, then we have $\underset{x_\nu}{E}[\nu < \pi] \in \mathfrak{E}.\mathfrak{Aq}(A)$, which on the basis of Def. 3a, b gives the condition (α).

Accordingly the conditions (α)—(γ) are equivalent, q.e.d.

Further theorems deal with axiomatizable and unaxiomatizable systems.

THEOREM 26. $\mathfrak{S}.\mathfrak{A} = \underset{Cn(X)}{E}[X \in \mathfrak{E}.\mathfrak{P}(S)]$.

[Def. 3a, b, Th. 18a]

THEOREM 27. *In order that $A \in \mathfrak{S} - \mathfrak{A}$, it is necessary and sufficient that there exists a sequence of sets X_ν of the type ω which satisfies the following formulas:* (α) $X_\nu \in \mathfrak{S}$ *for every* $\nu < \omega$; (β) $X_\mu \subseteq X_\nu$ *and* $X_\mu \neq X_\nu$ *for* $\mu < \nu < \omega$; (γ) $A = \sum_{\nu<\omega} X_\nu$. *The formula* (α) *can be replaced by* (α′) $X_\nu \in \mathfrak{S}.\mathfrak{A}$.

Proof. A. First let us assume that

(1) $$A \in \mathfrak{S} - \mathfrak{A}.$$

By applying Ths. 16 and 25 (and taking into account Def. 1) we infer from (1) that a sequence of sentences x_ν of type ω exists which satisfies the formulas

(2) $$x_\nu \,\overline{\in}\, Cn\left(\underset{x_\lambda}{E}[\lambda < \nu]\right) \quad for \quad \nu < \omega$$

and

(3) $$\underset{x_\nu}{E}[\nu < \omega] \in \mathfrak{Aq}(A).$$

By Th. 18a from (1) and (3) we obtain

(4) $$\underset{x_\nu}{E}[\nu < \omega] \subseteq S \quad and \quad Cn\left(\underset{x_\nu}{E}[\nu < \omega]\right) = A.$$

We put

(5) $$X_\nu = Cn\left(\underset{x_\lambda}{E}[\lambda < \nu]\right) \quad for \quad \nu < \omega.$$

By Th. 26 it follows from (4) and (5) that

(6) $$X_\nu \in \mathfrak{S} \quad for \quad \nu < \omega$$

and even that

(6′) $$X_\nu \in \mathfrak{S}.\mathfrak{A} \quad for \quad \nu < \omega.$$

In accordance with Th. 1a and by reference to (4) we obtain $Cn\left(\underset{x_\lambda}{E}[\lambda < \mu]\right) \subseteq Cn\left(\underset{x_\lambda}{E}[\lambda < \nu]\right)$ and from this by (5)

(7) $$X_\mu \subseteq X_\nu \quad for \quad \mu < \nu < \omega.$$

From (4), (5), and Ax. 2 we obtain $x_\mu \in X_\nu$, but in view of (2) and (5) $x_\mu \,\overline{\in}\, X_\mu$ for $\mu < \nu < \omega$; consequently

(8) $$X_\mu \neq X_\nu \quad for \quad \mu < \nu < \omega.$$

Finally it is to be noted that the class $\mathfrak{K}'$ of sets of sentences $Y_\nu = \underset{x_\lambda}{E}[\lambda < \nu]$, where $\nu < \omega$, satisfies the hypothesis of Cor. 5, whence $Cn\left(\sum_{Y \in \mathfrak{K}'} Y\right) = \sum_{Y \in \mathfrak{K}'} Cn(Y)$; on the basis of (4) and (5) we thus have

(9) $$A = \sum_{\nu<\omega} X_\nu.$$

From (6)–(9) (as well as (6′)) it results that

(10) *the sequences of sets X_ν of type ω satisfies the formulas* (α)–(γ) *(as well as the formula* (α′)) *of the theorem.*

B. Consider now a sequence of sets X_ν of the kind described in (10) (with the reference to (α′) omitted). Put

(11) $$\mathfrak{K} = \underset{X_\nu}{E}[\nu < \omega].$$

From (10) and (11) it follows at once that

(12) $$A = \sum_{X \in \mathfrak{K}} X$$

and that

(13) *the class $\mathfrak{K}$ satisfies condition* (α) *of Cor.* 5.

Since further by (10) and (11) $\mathfrak{K} \subseteq \mathfrak{S}$, we infer from (13) and (12) by means of Th. 12 that

(14) $$A = \sum_{X \in \mathfrak{K}} X \in \mathfrak{S}.$$

If it were the case that $\mathfrak{K}.\mathfrak{Aq}\left(\sum_{X\in\mathfrak{K}} X\right) \neq 0$, e.g. if

$$Y \in \mathfrak{K}.\mathfrak{Aq}\left(\sum_{X\in\mathfrak{K}} X\right),$$

then by (10) and (14) we should have $Y \in \mathfrak{S}.\mathfrak{Aq}(A)$ and $A \in \mathfrak{S}$, which by Th. 18 *c* gives $Y = A \in \mathfrak{K}$; but the last formula is in contradiction with (10) and (11) (since the sum of all terms of an increasing sequence of sets of type ω contains every term as a *proper* part). Consequently

(15) $$\mathfrak{K}.\mathfrak{Aq}\left(\sum_{X\in\mathfrak{K}} X\right) = 0.$$

By the use of Th. 23 we infer from (13)–(15) that

$$A = \sum_{X\in\mathfrak{K}} X \,\overline{\in}\, \mathfrak{A};$$

by combining this formula with (14) we at once obtain (1).

We have thus shown that *the existence of a sequence of sets described in* (10) *forms a necessary and sufficient condition for formula* (1), q.e.d.

From the last two theorems the following corollary is easily obtained:

Corollary 28. $\overline{\overline{\mathfrak{S}.\mathfrak{A}}} \leqslant \aleph_0$.

Proof. According to a well-known theorem of set theory Ax. 1 has the consequence that the class $\mathfrak{E}.\mathfrak{P}(S)$ is at most denumerable. By Th. 26 the function *Cn* establishes a many-to-one correlation between the members of this class and those of the class $\mathfrak{S}.\mathfrak{A}$; accordingly this latter class is also at most denumerable, $\overline{\overline{\mathfrak{S}.\mathfrak{A}}} \leqslant \aleph_0$, q.e.d.

Corollary 29. (*a*) *If* $A \in \mathfrak{S}-\mathfrak{A}$, *then* $\overline{\overline{\mathfrak{P}(A).\mathfrak{S}.\mathfrak{A}}} = \aleph_0$.

(*b*) *If* $\mathfrak{S}-\mathfrak{A} \neq 0$ (*or* $\mathfrak{P}(S)-\mathfrak{A} \neq 0$), *then* $\overline{\overline{\mathfrak{S}.\mathfrak{A}}} = \aleph_0$.

Proof. (*a*) By Th. 27 the hypothesis gives $\overline{\overline{\mathfrak{P}(A).\mathfrak{S}.\mathfrak{A}}} \geqslant \aleph_0$; since the inverse inequality follows immediately from Cor. 28, we finally have $\overline{\overline{\mathfrak{P}(A).\mathfrak{S}.\mathfrak{A}}} = \aleph_0$;

(*b*) results directly from (*a*) and Cor. 28 (with the help of Ths. 18 *a* and 20 *c* there is no difficulty in showing that the second premiss, $\mathfrak{P}(S)-\mathfrak{A} \neq 0$, implies the first, $\mathfrak{S}-\mathfrak{A} \neq 0$).

Theorem 30. *Either* $1 \leqslant \overline{\overline{\mathfrak{S}}} \leqslant \aleph_0$ *or* $\aleph_0 \leqslant \overline{\overline{\mathfrak{S}}} \leqslant 2^{\aleph_0}$.

Proof. According to the well-known theorem on the cardinal number of the power set we first infer from Ax. 1 and Def. 1 that $\overline{\overline{\mathfrak{S}}} \leqslant \overline{\overline{\mathfrak{P}(S)}} \leqslant 2^{\aleph_0}$; and by virtue of Th. 10 we also have $\overline{\overline{\mathfrak{S}}} \geqslant 1$. If now $\mathfrak{S} \subseteq \mathfrak{A}$ (or, what amounts to the same thing, $\mathfrak{S} = \mathfrak{S}.\mathfrak{A}$), then by Cor. 28 we have

$$1 \leqslant \overline{\overline{\mathfrak{S}}} \leqslant \aleph_0;$$

if, however, $\mathfrak{S}-\mathfrak{A} \neq 0$, then it follows from Cor. 29 *b* that

$$\aleph_0 \leqslant \overline{\overline{\mathfrak{S}}} \leqslant 2^{\aleph_0}, \text{ q.e.d.}$$

It is to be noted that the theorem above can be established without the use of the axiom of choice (otherwise the result would be quite trivial); the same applies also to the generalization of this theorem given below in Cor. 65. If we restrict ourselves to the consideration of deductive disciplines which presuppose sentential calculus, we can improve Th. 30 by showing that the class $\mathfrak{S}$ is either of the power 2^{ν} for some $\nu < \omega$ or else of the power $2^{\aleph_0}$.†

§ 5. Independent Sets of Sentences; Basis of a Set of Sentences

A set of sentences is called *independent* if it is not equivalent to any of its proper subsets. The class of all independent sets of sentences is denoted by '$\mathfrak{U}$':

Definition 4. $\mathfrak{U} = \underset{X}{E}[\mathfrak{P}(X).\mathfrak{Aq}(X) = \{X\}]$.

Some equivalent transformations of the above definition are given in the next two theorems.

Theorem 31. *The following conditions are equivalent:* (α) $A \in \mathfrak{U}$; (β) $A.Cn(X) \subseteq X \subseteq S$ *for every set* $X \subseteq A$; (γ) *the formulas* $X+Y \subseteq A$ *and* $Cn(X) = Cn(Y)$ *always imply* $X = Y$, *and in addition* $A \subseteq S$.

Proof. A. First we suppose that

(1) $$A \in \mathfrak{U}.$$

By Defs. 4 and 2 it follows that:

(2) $$A \subseteq S.$$

† See XII, Ths. 37 and 38 (p. 367).

Let X be any subset of A.

By the use of Th. 2*a* and with the help of (2) we obtain:

$$Cn((A-Cn(X))+X) = Cn((A-Cn(X))+Cn(X))$$
$$= Cn(A+Cn(X)) = Cn(A+X) = Cn(A),$$

and accordingly by Def. 2, $(A-Cn(X))+X \in \mathfrak{P}(A).\mathfrak{A}\mathfrak{q}(A)$. From this, by virtue of Def. 4 and with the help of (1), we infer that $(A-Cn(X))+X = A$ and consequently $A.(Cn(X)-X) = 0$. Thus by (2) we have

(3) $A.Cn(X) \subseteq X \subseteq S$ *for every set* $X \subseteq A$.

B. We assume next the condition (3). We consider any two sets X and Y which satisfy the formulas $X+Y \subseteq A$ and $Cn(X) = Cn(Y)$. It follows from (3) that

$$A.Cn(Y) = A.Cn(X) \subseteq X;$$

since further (3) gives the inclusion (2), we have by Ax. 2

$$Y \subseteq A.Cn(Y).$$

Consequently $Y \subseteq X$. In an exactly analogous manner we reach the inverse inclusion $X \subseteq Y$, so that finally $X = Y$. We have thus shown that

(4) *the formulas* $X+Y \subseteq A$ *and* $Cn(X) = Cn(Y)$ *always imply* $X = Y$, *and in addition* (2) *holds*.

C. Finally, let (4) be given. According to Def. 2 every set $X \in \mathfrak{P}(A).\mathfrak{A}\mathfrak{q}(A)$ satisfies the formula $Cn(X) = Cn(A)$; hence, if in (4) we put $Y = A$, we obtain $X = A$. Thus we have $\mathfrak{P}(A).\mathfrak{A}\mathfrak{q}(A) \subseteq \{A\}$; but since by Th. 13*a* the inclusion

$$\{A\} \subseteq \mathfrak{P}(A).\mathfrak{A}\mathfrak{q}(A)$$

also holds, we reach the identity $\mathfrak{P}(A).\mathfrak{A}\mathfrak{q}(A) = \{A\}$, which by Def. 4 gives the formula (1).

According to the above argument, (3) follows from (1), (4) from (3), and (1) from (4). *The conditions* (1), (3), *and* (4) *are thus equivalent*, q.e.d.

THEOREM 32. *In order that* $A \in \mathfrak{U}$, *it is necessary and sufficient that* $x \in S-Cn(A-\{x\})$, *for every* $x \in A$.

Proof. A. Let us assume that

(1) $A \in \mathfrak{U}$,

and apply Th. 31. According to the condition (γ) of this theorem it follows directly from (1) that

(2) $A \subseteq S$.

By putting $X = A-\{x\}$ in condition (β) of the same theorem we further obtain $A.Cn(A-\{x\}) \subseteq A-\{x\}$, whence

$$\{x\}.A.Cn(A-\{x\}) = 0.$$

Under the assumption that $x \in A$ the last formula gives $\{x\}.Cn(A-\{x\}) = 0$ and finally, by virtue of (2),

(3) $x \in S-Cn(A-\{x\})$ *for every* $x \in A$.

B. We now assume formula (3) and note initially that from this the inclusion (2) immediately follows. Let us assume that there exists a set $X \subseteq A$ which satisfies the formula

$$A.Cn(X)-X \neq 0.$$

Let for example $x \in A.Cn(X)-X$ and accordingly $X \subseteq A-\{x\}$, which by Th. 1*a* gives the inclusion $Cn(X) \subseteq Cn(A-\{x\})$; since then $x \in A.Cn(X)$, we infer at once that $x \in Cn(A-\{x\})$, which contradicts formula (3). Our assumption is thus disproved; consequently, in view of (2), we must assume that

(4) $A.Cn(X) \subseteq X \subseteq S$ *for every set* $X \subseteq A$.

By Th. 31 formula (1) follows from (4).

We have thus established the equivalence of the formulas (1) and (3) and thus proved the theorem.

From the last theorem we obtain immediately

COROLLARY 33. (*a*) $0 \in \mathfrak{U}$; [Th. 32]

(*b*) *in order that* $\{x\} \in \mathfrak{U}$, *it is necessary and sufficient that*

$$x \in S-Cn(0). \quad \text{[Th. 32]}$$

THEOREM 34. *If* $A \in \mathfrak{U}$, *then* $\mathfrak{P}(A) \subseteq \mathfrak{U}$. [Th. 31]

THEOREM 35. *If* $A \in \mathfrak{U}$, *then* $\mathfrak{P}(A).\mathfrak{A} = \mathfrak{P}(A).\mathfrak{E}$.

Proof. According to Th. 32 (or else 31) the hypothesis gives

(1) $A \subseteq S$.

Consequently we have $\mathfrak{P}(A).\mathfrak{E} \subseteq \mathfrak{P}(S).\mathfrak{E}$, from which by Th. 19$b$ the inclusion

(2) $$\mathfrak{P}(A).\mathfrak{E} \subseteq \mathfrak{P}(A).\mathfrak{A}$$

follows.

On the other hand let us assume that

(3) $$\mathfrak{P}(A).\mathfrak{A}-\mathfrak{E} \neq 0,$$

and accordingly let

(4) $$X \in \mathfrak{P}(A).\mathfrak{A}-\mathfrak{E}.$$

With the help of Ax. 1 we easily infer from (1)–(4) that the set X is denumerable; accordingly all elements of this set can be ordered in an infinite sequence of the type ω, with all terms distinct, such that

(5) $$X = \underset{x_\nu}{E}[\nu < \omega] \quad \textit{where } x_\mu \neq x_\nu \textit{ for } \mu < \nu < \omega.$$

According to Th. 34 it follows from (4) and the hypothesis that $X \in \mathfrak{U}$; from this by Th. 32 and with the help of (5) we obtain

(6) $$x_\nu \in S - Cn(X - \{x_\nu\}) \quad \textit{for every } \nu < \omega.$$

From (5) and (6) we further obtain

$$\underset{x_\mu}{E}[\mu < \nu] \subseteq X - \{x_\nu\} \subseteq S \quad \textit{for } \nu < \omega.$$

By Th. 1a this implies the inclusion

$$Cn\Big(\underset{x_\mu}{E}[\mu < \nu]\Big) \subseteq Cn(X - \{x_\nu\});$$

hence by (6) it follows that

(7) $$x_\nu \in S - Cn\Big(\underset{x_\mu}{E}[\mu < \nu]\Big) \quad \textit{for} \quad \nu < \omega.$$

By combining (5) and (7) with Th. 24 we easily obtain

$$X = \underset{x_\nu}{E}[\nu < \omega] \,\bar{\in}\, \mathfrak{A},$$

contrary to formula (4).

With this, assumption (3) is disproved and therefore we have

(8) $$\mathfrak{P}(A).\mathfrak{A} \subseteq \mathfrak{P}(A).\mathfrak{E}.$$

Inclusions (2) and (8) at once give the required identity

$$\mathfrak{P}(A).\mathfrak{A} = \mathfrak{P}(A).\mathfrak{E}, \quad \text{q.e.d.}$$

THEOREM 36. *If* $A \in \mathfrak{U}-\mathfrak{E}$, *then* $\overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}.\mathfrak{A}}} = \aleph_0$, *but* $\overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}}} = \overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}-\mathfrak{A}}} = 2^{\aleph_0}$.

Proof. By Ths. 32 and 35 as well as Ax. 1 we infer from the hypothesis that

(1) $$A \subseteq S, \qquad A \,\bar{\in}\, \mathfrak{A},$$

and

(2) $$\overline{\overline{A}} = \aleph_0.$$

According to Th. 18a it follows from (1) that $Cn(A) \in \mathfrak{S}$ and $A \in \mathfrak{Aq}(Cn(A))$; from this by Th. 20$c$ we obtain $Cn(A) \in \mathfrak{S}-\mathfrak{A}$. Hence, as a consequence of Cor. 29a, we obtain at once

(3) $$\overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}.\mathfrak{A}}} = \aleph_0.$$

Since by the hypothesis the set A is independent, we can apply Th. 31. Condition (γ) of this theorem asserts that the function Cn maps the class $\mathfrak{P}(A)$ on $\underset{Cn(X)}{E}[X \subseteq A]$ in one-to-one fashion; consequently the two classes have the same cardinal number. By a well-known theorem it follows from (2) that $\overline{\overline{\mathfrak{P}(A)}} = 2^{\aleph_0}$; hence

(4) $$\overline{\overline{\underset{Cn(X)}{E}[X \subseteq A]}} = 2^{\aleph_0}.$$

By means of Ths. 1a and 9a, using (1) we obtain without difficulty that $\underset{Cn(X)}{E}[X \subseteq A] \subseteq \mathfrak{P}(Cn(A)).\mathfrak{S}$; by combining this formula with (4) we get $\overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}}} \geqslant 2^{\aleph_0}$. But since by Th. 30 we also have $\overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}}} \leqslant 2^{\aleph_0}$, we finally obtain

(5) $$\overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}}} = 2^{\aleph_0}.$$

Formulas (3) and (5) at once give

(6) $$\overline{\overline{\mathfrak{P}(Cn(A)).\mathfrak{S}-\mathfrak{A}}} = 2^{\aleph_0} - \aleph_0 = 2^{\aleph_0}.$$

According to (3), (5), and (6) the theorem is completely proved.

COROLLARY 37. *If* $\mathfrak{U}-\mathfrak{E} \neq 0$, *then* $\overline{\overline{\mathfrak{S}.\mathfrak{A}}} = \aleph_0$, *but*

$$\overline{\overline{\mathfrak{S}}} = \overline{\overline{\mathfrak{S}-\mathfrak{A}}} = 2^{\aleph_0}.$$

[Th. 36, Cor. 28, Th. 30]

It is to be noted that within almost all deductive disciplines, and in particular within the simplest of them—the sentential calculus—it has been found possible to construct a set of sentences which is both infinite and independent, and thus to realize the hypothesis of the last corollary. Hence, it turns out that in all these disciplines there are more unaxiomatizable than axiomatizable systems; the deductive systems are, so to speak, as a rule unaxiomatizable, although in practice we deal almost exclusively with axiomatizable systems.‡ This paradoxical circumstance was first noticed by Lindenbaum in application to the sentential calculus.[1]

Every independent set of sentences which is equivalent to a given set A is called a *basis of the set A*; the class of all such sets of sentences is here denoted by '$\mathfrak{B}(A)$'.

DEFINITION 5. $\mathfrak{B}(A) = \mathfrak{A}\mathrm{q}(A).\mathfrak{U}$.

The above definition can be expressed otherwise as follows:

THEOREM 38. *In order that $B \in \mathfrak{B}(A)$, it is necessary and sufficient that $\mathfrak{P}(B).\mathfrak{A}\mathrm{q}(A) = \{B\}$.* [Defs. 5, 4, Th. 13 *b*]

In the usual terminology of set theory a set X is said to be *irreducible with respect to a class $\mathfrak{K}$ of sets* when $\mathfrak{P}(X).\mathfrak{K} = \{X\}$.[2] Thus the theorem above asserts that 'a basis of the set A of sentences' means the same as 'a set irreducible with respect to the class $\mathfrak{A}\mathrm{q}(A)$'.

Other properties of this concept are expressed in the following theorems:

THEOREM 39. *In order that $A \in \mathfrak{B}(A)$, it is necessary and sufficient that $A \in \mathfrak{U}$.* [Defs. 5, 4]

THEOREM 40. (*a*) *If $\mathfrak{B}(A).\mathfrak{B}(B) \neq 0$, then $A \in \mathfrak{A}\mathrm{q}(B)$;* [Def. 5, Th. 13 *b*]

(*b*) *if $A \in \mathfrak{A}\mathrm{q}(B)$, then $\mathfrak{B}(A) = \mathfrak{B}(B)$.* [Th. 13 *b*, Def. 5]

THEOREM 41. $\mathfrak{E}.\mathfrak{B}(A) = \mathfrak{U}.\mathfrak{A}\mathfrak{x}(A)$. [Defs. 5, 4, 3 *a*]

THEOREM 42. *If $A \in \mathfrak{A}$, then $\mathfrak{P}(A).\mathfrak{E}.\mathfrak{B}(A) \neq 0$.*

[1] Cf. IV, Th. 27 (p. 51).
[2] Cf. Tarski, A. (71), p. 48 (Def. 1).

‡ The last remark ("although in practice . . .") does not correspond to the present situation in the foundations of mathematics.

Proof. By Th. 22 and Def. 3 *a* the hypothesis implies that a set X exists which satisfies the formulas

(1) $$X \subseteq A,$$

(2) $$X \in \mathfrak{E} \quad \textit{and} \quad X \in \mathfrak{A}\mathrm{q}(A).$$

By Th. 13 *a*, *b* it results from (2) that $X \in \mathfrak{A}\mathrm{q}(X)$; consequently the class $\mathfrak{K} = \mathfrak{P}(X).\mathfrak{A}\mathrm{q}(X)$, which consists of subsets of the finite set X, is distinct from 0. Hence, by a familiar definition of finite sets,[1] we conclude that this class contains among its elements at least one irreducible set Y (i.e. a set with the property $\mathfrak{P}(Y).\mathfrak{K} = \{Y\}$). By Th. 38 this set Y forms a basis of X, and thus we have

(3) $$Y \subseteq X \quad \textit{and} \quad Y \in \mathfrak{B}(X).$$

From (1)–(3) we obtain at once

(4) $$Y \subseteq A \quad \textit{and} \quad Y \in \mathfrak{E}.$$

By Th. 40 *b* it follows from (2) that $\mathfrak{B}(A) = \mathfrak{B}(X)$, whence by (3)

(5) $$Y \in \mathfrak{B}(A).$$

Formulas (4) and (5) give at once

$$\mathfrak{P}(A).\mathfrak{E}.\mathfrak{B}(A) \neq 0, \quad \text{q.e.d.}$$

THEOREM 43. *If $A \in \mathfrak{A}$, then $\mathfrak{B}(A) \subseteq \mathfrak{E}$.*

Proof. If the conclusion were false, we should have, by Def. 5, $\mathfrak{A}\mathrm{q}(A).\mathfrak{U}-\mathfrak{E} \neq 0$. Accordingly let $X \in \mathfrak{A}\mathrm{q}(A)$ and $X \in \mathfrak{U}-\mathfrak{E}$. By Th. 35 the second of these formulas yields $X \overline{\in} \mathfrak{A}$; and hence, with the help of the first formula by using Th. 20 *c* we infer that $A \overline{\in} \mathfrak{A}$, contrary to the hypothesis.

We thus have $\mathfrak{B}(A) \subseteq \mathfrak{E}$, and so the theorem is proved.

COROLLARY 44. *The following conditions are equivalent:*

(α) $A \in \mathfrak{A}$; (β) $\mathfrak{B}(A) \neq 0$ *and* $\mathfrak{B}(A) \subseteq \mathfrak{E}$; ($\gamma$) $\mathfrak{E}.\mathfrak{B}(A) \neq 0$.

Proof. By Ths. 42 and 43, (β) is at once obtainable from (α); (γ) results immediately from (β); finally, on the basis of Th. 41 and Def. 3 *b*, (α) follows from (γ). Accordingly the formulas (α)–(γ) are equivalent, q.e.d.

[1] Cf. Tarski, A. (71), p. 49 (Def. 3).

By the corollary just established (or by Th. 42) every axiomatizable set of sentences possesses at least one basis. This result cannot be extended to unaxiomatizable sets of sentences on the basis of the axioms underlying this discussion[1] (it can only be shown that every set of sentences possesses an ordered basis; see remarks following Th. 16). On the other hand, for those deductive disciplines which presuppose the sentential calculus it can be proved that every set of sentences has a basis.†

§ 6. Consistent Sets of Sentences

The concepts of consistency and completeness, with which we are concerned in this and the next sections, are among the most important concepts of metamathematics; around these concepts is centred the research which is carried on today within special metadisciplines.

A set of sentences is called consistent if it is not equivalent to the set of all meaningful sentences (or, in other words, if the set of its consequences does not contain as elements all meaningful sentences).

According to the usual definition, a set of sentences is called consistent if there is no sentence which together with its negation belongs to the consequences of this set. Our definition thus diverges from the usual one, and indeed it has a much more general character since knowledge of the concept of negation is not presupposed; in consequence this definition can be applied even to those deductive disciplines in which the negation concept is either entirely lacking or at least does not exhibit the properties usually ascribed to it.[2] However, the two definitions of consistency prove to be equivalent for all those disciplines which are based upon the ordinary system of sentential calculus.

[1] For example, in the sentential calculus it is easy to construct sets of sentences without a basis. (A proof of this result appears in Schröter, K. (62 *a*), pp. 299–301.)

[2] Such a discipline, the so-called restricted sentential calculus is treated in IV, § 4.

† Cf. the remarks following Th. 25 in XII (p. 362).

We owe the idea of the definition adopted here to E. Post: in his investigations on the sentential calculus he has made use of a closely related definition.[1]

Denoting the class of all consistent sets of sentences by '$\mathfrak{W}$' we reach

Definition 6. $\mathfrak{W} = \mathfrak{P}(S) - \mathfrak{A}\mathfrak{q}(S)$.

The content of this definition is more clearly formulated in the following:

Theorem 45. *In order that $A \in \mathfrak{W}$, it is necessary and sufficient that $A \subseteq S$ and $Cn(A) \neq S$.* [Def. 6, Ths. 10, 18 *a*]

The following elementary properties of consistent sets of sentences deserve attention:

Theorem 46. (*a*) *If $A \in \mathfrak{W}$, then $\mathfrak{P}(A) \subseteq \mathfrak{W}$;* [Def. 6, Th. 14]

(*b*) *if $A \in \mathfrak{W}$, then $\mathfrak{A}\mathfrak{q}(A) \subseteq \mathfrak{W}$.* [Th. 45, Def. 2]

Theorem 47. $\mathfrak{S} - \mathfrak{W} = \{S\}$. [Def. 6, Ths. 7, 18 *c*]

Theorem 48. *Let $S \in \mathfrak{A}$; in order that* (α) *$A \in \mathfrak{W}$, it is necessary and sufficient that* (β) $\mathfrak{P}(A) . \mathfrak{E} \subseteq \mathfrak{W}$.

Proof. According to Th. 46 *a* the condition (β) is necessary for (α) to hold; it remains only to show that this condition is also sufficient.

Let us assume (β) and suppose that nevertheless

(1) $$A \bar{\in} \mathfrak{W}.$$

By Def. 6 it follows from (β) that $\mathfrak{P}(A) . \mathfrak{E} \subseteq \mathfrak{P}(S)$; from this we easily obtain:

(2) $$A \subseteq S.$$

By another application of Def. 6 we obtain from (1) and (2) $A \in \mathfrak{A}\mathfrak{q}(S)$. Since by hypothesis $S \in \mathfrak{A}$ holds, we infer by Th. 20 *c* that $A \in \mathfrak{A}$, whence by Th. 22 the formula $\mathfrak{P}(A) . \mathfrak{A}\mathfrak{x}(A) \neq 0$ follows. Accordingly by Def. 3 there exists a set X which satisfies the formulas

(3) $$X \in \mathfrak{P}(A) . \mathfrak{E}$$

[1] Post, E. L. (60), p. 177.

and

(4) $$X \in \mathfrak{A}\mathrm{q}(A).$$

From (β) and (3) it follows at once that $X \in \mathfrak{W}$; according to Th. 13*b*, (4) is equivalent to the formula $A \in \mathfrak{A}\mathrm{q}(X)$. Consequently by virtue of Th. 46*b* we obtain the formula $A \in \mathfrak{W}$, which coincides with (α) and contradicts the assumption (1).

We have thus shown that (β) is a sufficient condition for (α), and with this the proof is complete.

THEOREM 49. *Let* $S \in \mathfrak{A}$; *if the class* $\mathfrak{K}$ *satisfies the condition* (α) *of Th.* 4 (*or of Cor.* 5) *and if at the same time* $\mathfrak{K} \subseteq \mathfrak{W}$ *holds, then* $\sum_{X \in \mathfrak{K}} X \in \mathfrak{W}$.

Proof. Consider an arbitrary set $Y \in \mathfrak{E}.\mathfrak{P}\Big(\sum_{X \in \mathfrak{K}} X\Big)$. By virtue of condition (α) of Th. 4 we infer without difficulty that there is a set Z which satisfies the formulas $Z \in \mathfrak{K}$ and $Y \subseteq Z$ (cf. the proof of Th. 23). On the basis of the hypothesis the first formula yields $Z \in \mathfrak{W}$; hence by using Th. 46*a* and with the help of the second formula we obtain $Y \in \mathfrak{W}$.

It is thus proved that the formula $Y \in \mathfrak{E}.\mathfrak{P}\Big(\sum_{X \in \mathfrak{K}} X\Big)$ always implies $Y \in \mathfrak{W}$. Consequently we have the inclusion

$$\mathfrak{E}.\mathfrak{P}\Big(\sum_{X \in \mathfrak{K}} X\Big) \subseteq \mathfrak{W};$$

in accordance with the hypothesis $S \in \mathfrak{A}$ and with the help of Th. 48 we get from this at once

$$\sum_{X \in \mathfrak{K}} X \in \mathfrak{W}, \quad \text{q.e.d.}$$

The formula $S \in \mathfrak{A}$, which occurs as a premiss in the two preceding theorems, as well as in some later ones (Ths. 51, 56, and 57), cannot be derived from the axioms listed in § 1. It is, however, satisfied for all known formalized disciplines; even the following logically stronger assertion holds: *there is a sentence* $x \in S$ *such that* $Cn(\{x\}) = S$. Nevertheless it does not seem desirable to include this formula among the axioms, on account of its special and, in a certain sense, accidental character. (However, the italicized sentence was added as a special axiom, Ax. 5, in article III, p. 31.)

§ 7. Decision Domain of a Set of Sentences, Complete Sets of Sentences

By the *decision domain* of the set A of sentences we understand the set of all sentences which are either consequences of A or which, when added to A, yield an inconsistent set of sentences. A set of sentences is said to be *complete* or *absolutely complete* if its decision domain contains all meaningful sentences.[1]

With regard to this definition of completeness we can repeat all remarks which were made above in § 6 in connexion with the definition of consistency. This definition is also due to Post, who has used it in his investigations into the sentential calculus.[2] As is easily seen the usual definition of completeness, resting upon the concept of negation, is quite unsuitable for the sentential calculus (as well as for all disciplines which contain sentences with so-called free variables). In fact, by the usual definition, a set of sentences is called complete if, for every sentence, either the sentence itself or its negation belongs to the consequences of this set; and, as is well known, the ordinary system of sentential calculus is complete in the sense that its decision domain contains all meaningful sentences, although it is not complete according to the usual definition. The two definitions prove to be equivalent for all those disciplines which presuppose sentential calculus (and in which no expression containing free variables is regarded as a sentence).

The concept of absolute completeness is of great importance for those metadisciplines in which the objects of investigation are 'poor', elementary disciplines of an uncomplicated logical structure (e.g. the sentential calculus, or disciplines without predicate variables).[3] On the other hand this concept has not yet played an important part in investigations on 'rich', logically more complicated disciplines (e.g. the system of *Principia Mathematica*). The cause of this is perhaps to be sought in the

[1] The concept of completeness also occurs with another meaning, discussed, for instance, in Fraenkel, A. (16), § 18.4, pp. 347–54; the meaning used here corresponds to Fraenkel's *Entscheidungsdefinitheit*.

[2] See Post, E. L. (60), p. 177.

[3] Cf. Langford, C. H. (44), p. 115, as well as IV and Presburger, M. (61). For later developments in this domain consult Tarski, A. (84).

widespread, perhaps intuitively plausible, but not always strictly founded, belief in the incompleteness of all systems developed within these disciplines and known at the present day.[1] It is nevertheless to be expected that the concept of completeness will some day attain a greater importance even for the latter disciplines. For, although all known consistent systems of this kind may be incomplete, yet Th. 56 to be established below offers at least a theoretical possibility of extending every such system to one which is both consistent and complete. The question now arises how this extension is to be carried out so as to be 'effective', as natural as possible, and at the same time agreeing with some philosophical viewpoint.[2]

Moreover, it is to be noted that some more special problems which are closely connected with the concept of completeness have already been successfully investigated even with respect to those 'rich' disciplines. The investigations referred to have aimed at and succeeded in proving that all meaningful sentences of a particular logical form (e.g. all sentences without function variables) belong to the decision domain of some given system. It might be useful for such investigations to introduce the concept of relative completeness: a set A of sentences is said to be *relatively complete with respect to the set B of sentences* if the decision domain of A includes the set B; this concept will not be discussed further here.[3]

The decision domain of the set A of sentences will be denoted by '$\mathfrak{Ent}(A)$' and the class of all complete sets of sentences by '$\mathfrak{B}$'.

DEFINITION 7. (*a*) $\mathfrak{Ent}(A) = Cn(A)+S.\underset{x}{E}[A+\{x\} \overline{\in} \mathfrak{W}]$.

(*b*) $\mathfrak{B} = \mathfrak{P}(S).\underset{X}{E}[\mathfrak{Ent}(X) = S]$.

[1] See Fraenkel, A. (16), pp. 347–54.

[2] Since this article first appeared in print, several important contributions concerned with the concept of completeness have been published, throwing much light on the questions discussed in the last paragraph. See in the first place Gödel, K. (22).

[3] The investigations on absolute completeness in a 'poor' discipline are as a rule equivalent to those on relative completeness in a more extensive discipline. For this reason the works cited on p. 93, note 3 can be regarded as examples of investigations on relative completeness, if they are related to an extended discipline (e.g. to *Principia Mathematica*).

The following elementary properties of decision domains should be noted:

THEOREM 50. (*a*) *If* $A \subseteq S$, *then* $A \subseteq \mathfrak{Ent}(A) \subseteq S$. [Ax. 2, Def. 7 *a*]

(*b*) *If* $A \subseteq B \subseteq S$, *then* $\mathfrak{Ent}(A) \subseteq \mathfrak{Ent}(B)$. [Ths. 1 *a*, 46 *a*, Def. 7 *a*]

(*c*) *If* $A \in \mathfrak{Aq}(B)$, *then* $\mathfrak{Ent}(A) = \mathfrak{Ent}(B)$. [Def. 2, Ths. 13 *b*, 15 *a*, 46 *b*, Def. 7 *a*]

THEOREM 51. *Let* $S \in \mathfrak{A}$; *if* $A \subseteq S$, *then*

$$\mathfrak{Ent}(A) = \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} \mathfrak{Ent}(X).$$

Proof. We consider a sentence $y \in S$ such that $A+\{y\} \overline{\in} \mathfrak{W}$. Hence, by Th. 48 and with the help of the hypothesis, we conclude that $\mathfrak{P}(A+\{y\}).\mathfrak{E}-\mathfrak{W} \neq 0$. Thus there is a set Y which satisfies the formulas $Y \in \mathfrak{P}(A+\{y\}).\mathfrak{E}$ and $Y \overline{\in} \mathfrak{W}$. The first formula gives us $Y-\{y\} \in \mathfrak{P}(A).\mathfrak{E}$, and with the help of Th. 46 *a* obtain from the second formula $Y+\{y\} = (Y-\{y\})+\{y\} \overline{\in} \mathfrak{W}$. Consequently there exists a set X such that $X \in \mathfrak{P}(A).\mathfrak{E}$ and $X+\{y\} \overline{\in} \mathfrak{W}$ (and in fact $X = Y-\{y\}$); thus we have

$$y \in \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} \Big(\underset{x}{E}[X+\{x\} \overline{\in} \mathfrak{W}]\Big).$$

From this argument it follows immediately that

$$(1) \qquad S.\underset{x}{E}[A+\{x\} \overline{\in} \mathfrak{W}] \subseteq \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} \Big(S.\underset{x}{E}[X+\{x\} \overline{\in} \mathfrak{W}]\Big).$$

According to Ax. 4 we have further

$$(2) \qquad Cn(A) = \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} Cn(X).$$

Formulas (1) and (2) give at once

$$Cn(A)+S.\underset{x}{E}[A+\{x\} \overline{\in} \mathfrak{W}] \subseteq \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} \Big(Cn(X)+S.\underset{x}{E}[X+\{x\} \overline{\in} \mathfrak{W}]\Big),$$

whence by Def. 7 *a*

$$(3) \qquad \mathfrak{Ent}(A) \subseteq \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} \mathfrak{Ent}(X).$$

On the other hand it follows from Th. 50 *b* that

$$\mathfrak{Ent}(X) \subseteq \mathfrak{Ent}(A)$$

for every set $X \in \mathfrak{P}(A).\mathfrak{E}$, whence

(4) $$\sum_{X \in \mathfrak{P}(A).\mathfrak{E}} \mathfrak{Ent}(X) \subseteq \mathfrak{Ent}(A).$$

Finally, from (3) and (4) we obtain

$$\mathfrak{Ent}(A) = \sum_{X \in \mathfrak{P}(A).\mathfrak{E}} \mathfrak{Ent}(X), \quad \text{q.e.d.}$$

Def. 7 *b* can be modified in various ways:

THEOREM 52. *The following conditions are equivalent:*

(α) $A \in \mathfrak{B}$; (β) $A \subseteq S$ *and* $\mathfrak{Q}(A).\mathfrak{W} \subseteq \mathfrak{Aq}(A)$; ($\gamma$) $A \subseteq S$ *and* $\mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W} \subseteq \{Cn(A)\}$; ($\delta$) $\mathfrak{Q}(A).\mathfrak{S} = \{S, Cn(A)\}$.

Proof. A. Let us assume formula (α) and consider any set $X \in \mathfrak{Q}(A).\mathfrak{W}$. Thus according to Def. 6 $A \subseteq X \subseteq S$; hence by Def. 7 *a*, *b* and with the help of (α) we infer that $X \subseteq \mathfrak{Ent}(A)$ and that consequently for every $x \in X$ either $x \in Cn(A)$ or $A+\{x\} \bar{\in} \mathfrak{W}$; but since $A+\{x\} \subseteq X$, it follows from Th. 46 *a* that $A+\{x\} \in \mathfrak{W}$ and therefore $x \in Cn(A)$ for every $x \in X$. From this it follows that $X \subseteq Cn(A)$ and further that

$$A \subseteq X \subseteq Cn(A);$$

on the basis of Th. 1 *a*, *b* this last formula gives

$$Cn(A) \subseteq Cn(X) \subseteq Cn(A), \quad \text{i.e.} \quad Cn(A) = Cn(X),$$

whence by Def. 2 the formula $X \in \mathfrak{Aq}(A)$ follows.

From these considerations we conclude at once that

$$\mathfrak{Q}(A).\mathfrak{W} \subseteq \mathfrak{Aq}(A);$$

by Def. 7 *b*, (α) gives also the inclusion $A \subseteq S$. It is thus proved that

(1) (β) *follows from* (α).

B. Further by Th. 18 *b* we have

(2) (β) *implies* (γ).

C. It is likewise easy by Ths. 47 and 9 *a* to establish that

(3) (δ) *follows from* (γ).

D. Now we assume that (δ) holds; from this it results immediately that $A \subseteq S$. Let further an arbitrary sentence $x \in S$ be given. By virtue of Th. 9 *a* we have

$$Cn(A+\{x\}) \in \mathfrak{Q}(A+\{x\}).\mathfrak{S}.$$

Thus *a fortiori* $Cn(A+\{x\}) \in \mathfrak{Q}(A).\mathfrak{S}$. Hence by ($\delta$) we obtain

$$Cn(A+\{x\}) \in \{S, Cn(A)\}.$$

If $Cn(A+\{x\}) = S$, then it follows from Th. 45 that $A+\{x\} \bar{\in} \mathfrak{W}$; but if

$$Cn(A+\{x\}) = Cn(A),$$

then, by Ax. 2, we have $x \in Cn(A+\{x\})$ and consequently $x \in Cn(A)$. Thus by Def. 7 *a* it follows in either case that

$$x \in \mathfrak{Ent}(A).$$

By means of this argument we reach the inclusion $S \subseteq \mathfrak{Ent}(A)$, which by Th. 50 *a* leads to the identity $\mathfrak{Ent}(A) = S$; and, in accordance with Def. 7 *b*, (α) is a consequence of this identity. Thus

(4) (α) *follows from* (δ).

By combining (1)–(4) we infer at once that *the conditions* (α)–(δ) *are equivalent*, q.e.d.

THEOREM 53. *The following conditions are equivalent:* (α) $A \in \mathfrak{S}.\mathfrak{W}.\mathfrak{B}$; ($\beta$) $A \in \mathfrak{W}$ *and for every* $x \in S$ *either* $x \in A$ *or* $A+\{x\} \bar{\in} \mathfrak{W}$; ($\gamma$) $\mathfrak{Q}(A).\mathfrak{W} = \{A\}$.

Proof. A. By comparing Def. 7 *a*, *b* and Th. 7 we easily see that

(1) (β) *follows from* (α).

B. Let us assume (β). We consider an arbitrary set

$$X \in \mathfrak{Q}(A).\mathfrak{W},$$

which thus, by Def. 6, satisfies the formula $A \subseteq X \subseteq S$. For every $x \in X$ we have $A+\{x\} \subseteq X$ and consequently, by Th. 46 *a* $A+\{x\} \in \mathfrak{W}$, which by virtue of (β) gives the formula $x \in A$; consequently we have $X \subseteq A$ and in fact $X = A$. Hence we conclude that $\mathfrak{Q}(A).\mathfrak{W} \subseteq \{A\}$; since the inverse inclusion follows directly from (β) we finally reach the formula (γ), i.e.

$$\mathfrak{Q}(A).\mathfrak{W} = \{A\}.$$

It is thus proved that

(2) (γ) *follows from* (β).

C. Next we assume that (γ) holds. From this it results immediately that $A \in \mathfrak{W}$ and therefore $A \subseteq S$. By Ths. 9*a* and 18*b* we thus have $Cn(A) \in \mathfrak{Q}(A).\mathfrak{S}.\mathfrak{A}\mathfrak{q}(A)$; hence, with the help of Th. 46*b*, we obtain $Cn(A) \in \mathfrak{Q}(A).\mathfrak{W}$. The comparison of the last two formulas with (γ) gives first

$$Cn(A) = A \in \mathfrak{S}$$

and secondly $\mathfrak{Q}(A).\mathfrak{W} \subseteq \mathfrak{A}\mathfrak{q}(A)$.‡

Now, since the condition (β) of Th. 52 is satisfied, we have also $A \in \mathfrak{V}$, so that finally the whole formula (α), $A \in \mathfrak{S}.\mathfrak{W}.\mathfrak{V}$, is derived. Accordingly we have

(3) (γ) *implies* (α).

From (1)–(3) it follows at once that *the conditions* (α)–(γ) *are equivalent.* This completes the proof.

THEOREM 54. (*a*) *If* $A \in \mathfrak{V}$, *then* $\mathfrak{Q}(A).\mathfrak{P}(S) \subseteq \mathfrak{V}$; [Def. 7*b*, Th. 50*a*, *b*]

(*b*) *if* $A \in \mathfrak{V}$, *then* $\mathfrak{A}\mathfrak{q}(A) \subseteq \mathfrak{V}$. [Defs. 2, 7*b*, Th. 50*c*]

THEOREM 55. (*a*) $\mathfrak{P}(S) = \mathfrak{W}+\mathfrak{V}$; [Th. 45, Defs. 6, 7*a*, *b*]

(*b*) $S \in \mathfrak{V}$. [Ths. 47, 55*a*]

The following interesting theorem, due to Lindenbaum, asserts that (provided the set S is axiomatizable) every consistent set of sentences can be extended to a consistent and complete system.

THEOREM 56. *Let* $S \in \mathfrak{A}$; *if* $A \in \mathfrak{W}$, *then*

$$\mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W}.\mathfrak{V} \neq 0.$$

Proof. According to Def. 6 and Th. 47 the hypothesis gives $A \subseteq S$ and $A \neq S$, whence $S \neq 0$. Thus in view of Ax. 1 all sentences can be ordered in an infinite sequence of type ω (not necessarily with all terms distinct), such that

(1) $$S = \underset{x_\lambda}{E}[\lambda < \omega].$$

Let

(2) λ_0 *be the smallest of the numbers* $\lambda < \omega$ *which satisfy the formula* $A+\{x_\lambda\} \in \mathfrak{W}$.

‡ Compare the last paragraph on p. 92 as well as the footnote in III, p. 34.

Further, let

(3) λ_ν (*where* $0 < \nu < \omega$) *be the smallest of the numbers* $\lambda < \omega$ *which satisfy the conditions*

$$A+\underset{x_{\lambda_\mu}}{E}[\mu < \nu]+\{x_\lambda\} \in \mathfrak{W} \quad \textit{and} \quad \lambda > \lambda_\mu \textit{ for every } \mu < \nu.$$

Finally, let

(4) π *be the smallest ordinal number with which the conditions* (2) *and* (3) *correlate no number* λ_π.

Now we put

(5) $$X_0 = A, \quad X_{\nu+1} = A+\underset{x_{\lambda_\mu}}{E}[\mu \leqslant \nu] \quad \textit{for} \quad \nu < \pi$$

and

(6) $$X = X_0+\sum_{\nu<\pi} X_{\nu+1}.$$

The conditions (2)–(4) give at once

(7) $$\pi \leqslant \omega.$$

From (4)–(6) we obtain

(8) $$A \subseteq X,$$

and in general

(9) $$X_0 \subseteq X_{\mu+1} \subseteq X_{\nu+1} \subseteq X \quad \textit{for} \quad \mu < \nu < \pi.$$

From (2)–(5) we infer without difficulty that

(10) $$X_0 \in \mathfrak{W} \quad \textit{and} \quad X_{\nu+1} \in \mathfrak{W} \quad \textit{for} \quad \nu < \pi.$$

Let $\mathfrak{K} = \{X_0\}+\underset{X_{\nu+1}}{E}[\nu < \pi]$. By (9) and (7) the class $\mathfrak{K}$ satisfies the condition (α) of Cor. 5; by (10) the inclusion $\mathfrak{K} \subseteq \mathfrak{W}$ holds. Taking into account the formula $S \in \mathfrak{A}$ assumed in the hypothesis we see that all premisses of Th. 49 are satisfied. Accordingly we have $\sum_{Y \in \mathfrak{K}} Y \in \mathfrak{W}$, or, by (6),

(11) $$X \in \mathfrak{W}.$$

From (4)–(6) we obtain the formula

(12) $$X = A+\underset{x_{\lambda_\mu}}{E}[\mu < \pi].$$

By an indirect argument (analogous to the one used in refuting the assumption (9) in the proof of Th. 16) we easily infer from

(2)–(5) that for every number $\lambda < \omega$ which is distinct from all numbers λ_μ with $\mu < \pi$ either $X_0+\{x_\nu\} \,\bar{\in}\, \mathfrak{W}$ holds or there is a number $\nu < \pi$ which satisfies the formula

$$X_{\nu+1}+\{x_\lambda\} \,\bar{\in}\, \mathfrak{W}.$$

By virtue of (9) and Th. 46*a* we have *a fortiori* $X+\{x_\nu\} \,\bar{\in}\, \mathfrak{W}$. Hence by (1) and (12) we obtain

(13) *for every* $x \in S$ *either* $x \in X$ *or* $X+\{x\} \,\bar{\in}\, \mathfrak{W}$.

According to (11) and (13) the set X satisfies the condition (β) of Th. 53 (for $A = X$); consequently,

(14) $$X \in \mathfrak{S}.\mathfrak{W}.\mathfrak{V}.$$

The formulas (8) and (14) at once give

$$\mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W}.\mathfrak{V} \neq 0, \quad \text{q.e.d.}$$

Besides the concept of completeness, and perhaps more often than this concept (especially with regard to logically more complicated and comprehensive disciplines), two other concepts are treated in metamathematics which are related in content to completeness: *non-ramifiability* and *categoricity* (monomorphy).[1] These concepts are not reducible to those of sentence and consequence.† The definition and the establishment of the fundamental properties and of the mutual connexions of these concepts, as well as the clarification of their relation to the concept of completeness, must be left to special metadisciplines.

§ 8. Cardinal and Ordinal Degree of Completeness

In order to obtain a classification and characterization of incomplete sets of sentences, we introduce here the concept of the degree of completeness of a set of sentences, and we do this in two ways, namely by correlating a cardinal number and an ordinal number with every set of sentences. The *cardinal degree*

[1] Information about these concepts is given in Fraenkel, A. (16), pp. 347–54, where the literature on the subject is also listed. For the definitions of these notions see also X, pp. 310–314 ff., and XIII, p. 390.

† Nor can they be defined in terms of those notions which are discussed in XII.

of completeness of the set A of sentences, symbolically $\mathfrak{g}(A)$, is the number of all systems which include the set A; the *ordinal degree of completeness*, in symbols $\gamma(A)$, is identical with the smallest ordinal number π for which there is no strictly increasing sequence of type π of consistent systems which include the set A. In formulas we thus have

Definition 8. (*a*) $\mathfrak{g}(A) = \overline{\overline{\mathfrak{Q}(A).\mathfrak{S}}}$;

(*b*) $\gamma(A)$ *is the smallest ordinal number* π *for which there is no sequence of type* π *of sets* X_ν *which satisfy the formulas:*

(α) $X_\nu \in \mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W}$ *for* $\nu < \pi$, *where* $A \subseteq S$,

and

(β) $X_\mu \subseteq X_\nu$ *and* $X_\mu \neq X_\nu$ *for* $\mu < \nu < \pi$.

Def. 8*b* may be transformed as follows:

Theorem 57. $\gamma(A)$ *is identical with the smallest ordinal number* π *for which there is no sequence of type* π *of sentences* x_ν *which satisfy the formula* (α) $x_\nu \in S - Cn\big(A + \underset{x_\mu}{E}[\mu < \nu]\big)$ *for* $\nu < \pi$, *where* $A \subseteq S$.

Proof. Let a number ξ be given such that

(1) $$\xi < \gamma(A).$$

By Def. 8*b* there is a sequence of sets X_ν of type ξ which satisfy the formulas:

(2) $X_\nu \in \mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W}$ *for* $\nu < \xi$, *where* $A \subseteq S$;

(3) $X_\mu \subseteq X_\nu$ *and* $X_\mu \neq X_\nu$ *for* $\mu < \nu < \xi$.

We put further

(4) $X_\xi = S$ (*in case* ξ *is not a limit number*).

By Ths. 45 and 47 the formulas (2) and (4) give

(5) $X_\nu \subseteq S = X_\xi$ *and* $X_\nu \neq S = X_\xi$ *for* $\nu < \xi$.

From (3) and (5) we infer at once that the sets $X_{\nu+1} - X_\nu$, where $\nu < \xi$, are distinct from 0; hence with every such set one of its elements x_ν can be correlated in such a way that

(6) $x_\nu \in X_{\nu+1} - X_\nu$ *for* $\nu < \xi$.

(This correlation does not require the axiom of choice since by

(5) and Ax. 1 all sets $X_{\nu+1}-X_\nu$ with $\nu < \xi$ are subsets of the at most denumerable set S.)

It follows from (2), (3), and (6) that $A+\underset{x_\mu}{E}[\mu < \nu] \subseteq X_\nu$ for every $\nu < \xi$; hence with the help of (2) and Th. 8 we obtain $Cn\big(A+\underset{x_\mu}{E}[\mu < \nu]\big) \subseteq X_\nu$. But since by (5) and (6) $x_\nu \in S-X_\nu$, we finally have

(7) $x_\nu \in S-Cn\big(A+\underset{x_\mu}{E}[\mu<\nu]\big)$ *for* $\nu < \xi$ *(where* $A \subseteq S$*).*

With this we have proved that (1) always implies (7): for every number $\xi < \gamma(A)$ a sequence of sentences of type ξ can be constructed which satisfies formula (α) of the theorem here discussed. But since by hypothesis there is no such sequence for the number π, the inequality $\pi < \gamma(A)$ cannot hold. Consequently,

(8) $$\pi \geqslant \gamma(A).$$

By an analogous argument the inverse inequality

(9) $$\pi \leqslant \gamma(A)$$

can be proved. For let us consider an arbitrary number $\xi < \pi$. According to the hypothesis there is a sequence of sentences x_ν of type ξ which satisfies the formula (7). By putting

$$X_\nu = Cn\big(A+\underset{x_\mu}{E}[\mu<\nu]\big) \text{ for } \nu < \xi,$$

we easily conclude from Ths. 9 *a*, 47, and 1 *a* that the sequence of sets X_ν satisfies the formulas (2) and (3). But by Def. 8 *b* no such sequence of sets can be constructed for the number $\gamma(A)$. Consequently the inequality $\gamma(A) < \pi$ cannot hold and we obtain the formula (9).

Formulas (8) and (9) at once give the required identity

$$\gamma(A) = \pi, \quad \text{q.e.d.}$$

THEOREM 58. (*a*) *If* $A \subseteq B$, *then*

$$\mathfrak{g}(A) \geqslant \mathfrak{g}(B) \quad \textit{and} \quad \gamma(A) \geqslant \gamma(B); \qquad \text{[Def. 8 } a, b]$$

(*b*) *if* $A \in \mathfrak{Aq}(B)$, *then* $\mathfrak{g}(A) = \mathfrak{g}(B)$ *and* $\gamma(A) = \gamma(B)$.

Proof of (*b*). By Def. 2 the hypothesis gives $Cn(A) = Cn(B)$. From this by Th. 8 we infer that for every set $X \in \mathfrak{S}$ the inclusions $A \subseteq X$ and $B \subseteq X$ are equivalent, whence

$$\mathfrak{Q}(A).\mathfrak{S} = \mathfrak{Q}(B).\mathfrak{S}.$$

By virtue of Def. 8 *a*, *b* we obtain at once from the last formula the two required identities:

$$\mathfrak{g}(A) = \mathfrak{g}(B) \quad \text{and} \quad \gamma(A) = \gamma(B).$$

It is left to the reader to establish the connexions between the numbers $\mathfrak{g}(A)$ and $\mathfrak{g}(B)$, or $\gamma(A)$ and $\gamma(B)$, under more special assumptions (e.g. $A \in \mathfrak{P}(B)-\mathfrak{Aq}(B)$).

In the following theorem a characterization is given of the most important categories of sets of sentences, in fact of consistent and complete sets, in terms of the notions just defined.

THEOREM 59. (*a*) *The formulas* $A \subseteq S$, $\mathfrak{g}(A) \geqslant 1$ *and* $\gamma(A) \geqslant 1$ *are equivalent*; [Def. 8 *a*, *b*, Th. 10][1]

(*b*) *the formulas* $A \in \mathfrak{B}$, $1 \leqslant \mathfrak{g}(A) \leqslant 2$, *and* $1 \leqslant \gamma(A) \leqslant 2$ *are equivalent*; [Def. 8 *a*, *b*, Ths. 52, 10, 47, 9 *a*, 46 *a*]

(*c*) *the formulas* $A \in \mathfrak{W}$, $\mathfrak{g}(A) \geqslant 2$, *and* $\gamma(A) \geqslant 2$ *are equivalent.* [Def. 8 *a*, *b*, Ths. 9 *a*, 10, 45, 47, 46 *a*]

The proof is quite elementary.

COROLLARY 60. (*a*) *The formulas*

$$A \in \mathfrak{Aq}(S) \ (\textit{or } A \in \mathfrak{P}(S)-\mathfrak{W}),\ \mathfrak{g}(A) = 1,\ \textit{and}\ \gamma(A) = 1$$

are equivalent; [Th. 59 *a*, *c*, Defs. 2, 6]

(*b*) *the formulas* $A \in \mathfrak{W}.\mathfrak{B}$, $\mathfrak{g}(A) = 2$, *and* $\gamma(A) = 2$ *are equivalent;* [Th. 59 *b*, *c*]

(*c*) *the formulas* $A \in \mathfrak{W}-\mathfrak{B}$, $\mathfrak{g}(A) \geqslant 3$, *and* $\gamma(A) \geqslant 3$ *are equivalent.* [Th. 59 *b*, *c*]

THEOREM 61. *For every set A we have*

$$\mathfrak{g}(A) \leqslant 2^{\aleph_0} \quad \textit{and} \quad \gamma(A) \leqslant \Omega.$$

Proof. By Def. 1 we have $\mathfrak{S} \subseteq \mathfrak{P}(S)$; hence by Def. 8 *a* it follows that $\mathfrak{g}(A) \leqslant \overline{\overline{\mathfrak{P}(S)}}$. On the other hand (by a well-known theorem on the cardinal number of the power set) Ax. 1 implies the formula $\overline{\overline{\mathfrak{P}(S)}} \leqslant 2^{\aleph_0}$. Thus we finally reach the required inequality $\mathfrak{g}(A) \leqslant 2^{\aleph_0}$.

If $\gamma(A) > \Omega$ were the case, then by Defs. 8 *b* and 6 there would exist an increasing sequence (without repeating terms) of the type Ω of subsets of the set S; but this contradicts Ax. 1

[1] It is assumed that under the hypothesis $A \subseteq S$ at least the empty sequence of type 0 satisfies (vacuously) the conditions (α) and (β) of Def. 8 *b*.

according to which the set S is at most denumerable. Consequently $\gamma(A) \leqslant \Omega$, and with that the proof is complete.

THEOREM 62. *If* $A \subseteq S$ *and if there is a set* X *such that* $X \overline{\in} \mathfrak{C}$ *and* $x \in S - Cn(A+(X-\{x\}))$ *for every* $x \in X$, *then*

$$\mathfrak{g}(A) = 2^{\aleph_0} \quad \textit{and} \quad \gamma(A) = \Omega.$$

Proof. Applying Ths. 7 and 32, the following is obtained from Cor. 37:

(1) *if there is a set* X *such that* $X \overline{\in} \mathfrak{C}$ *and* $x \in S - Cn(X-\{x\})$ *for every* $x \in X$, *then*

$$\overline{\overline{\underset{Y}{E}[Cn(Y) = Y \subseteq S]}} = 2^{\aleph_0}.$$

We put

(2) $$F(Y) = Cn(A+Y) \quad \textit{for every set } Y \subseteq S.$$

By the relativization theorem, Th. 6, all the axioms postulated in § 1 remain valid if everywhere in them 'Cn' is replaced by the symbol 'F' just defined (although the variables in the axioms must also be renamed). Therefore all consequences of these axioms also remain valid (cf. the remarks after Th. 6). This applies in particular to (1) which, by the transformation decribed above, and in view of (2), becomes

(3) *if there is a set* X *such that* $X \overline{\in} \mathfrak{C}$ *and* $x \in S - Cn(A+(X-\{x\}))$ *for every* $x \in X$, *then*

$$\overline{\overline{\underset{Y}{E}[Cn(A+Y) = Y \subseteq S]}} = 2^{\aleph_0}.$$

From (3) with the help of the hypothesis we obtain

(4) $$\overline{\overline{\underset{Y}{E}[Cn(A+Y) = Y \subseteq S]}} = 2^{\aleph_0}.$$

By means of Ths. 7 and 9 *a* it is now easy to show that the formulas $Cn(A+Y) = Y \subseteq S$ and $Y \in \mathfrak{Q}(A).\mathfrak{S}$ are equivalent. Accordingly it follows from (4) that $\overline{\overline{\mathfrak{Q}(A).\mathfrak{S}}} = 2^{\aleph_0}$; and hence by means of Def. 8 *a* we obtain immediately the required formula

(5) $$\mathfrak{g}(A) = 2^{\aleph_0}.$$

We now consider an arbitrary number ξ such that

(6) $$\xi < \Omega,$$

thus $\xi \leqslant \aleph_0$. By Ax. 1 the set $X \subseteq S$ involved in the hypothesis is countable. Hence a sequence of type ξ can be constructed consisting entirely of distinct sentences x_ν of the set X and thus satisfying the formulas:

(7) $$\underset{x_\nu}{E}[\nu < \xi] \subseteq X \subseteq S \quad \textit{and} \quad x_\mu \neq x_\nu \quad \textit{for} \quad \mu < \nu < \xi.$$

Let ν be an arbitrary number $< \xi$. From (7) we infer that $A + \underset{x_\mu}{E}[\mu < \nu] \subseteq A+(X-\{x_\nu\}) \subseteq S$, whence by Th. 1 *a* the inclusion

$$Cn\Big(A+\underset{x_\mu}{E}[\mu < \nu]\Big) \subseteq Cn(A+(X-\{x_\nu\}))$$

follows. But since by virtue of the hypothesis

$$x_\nu \in S - Cn(A+(X-\{x_\nu\})),$$

we have further

(8) $$x_\nu \in S - Cn\Big(A+\underset{x_\mu}{E}[\mu < \nu]\Big) \quad \textit{for every } \nu < \xi.$$

Thus (6) implies (8): for every number $\xi < \Omega$ there is a sequence of sentences of type ξ which satisfies the formula (8). But according to Th. 57 there exists no such sequence for the number $\gamma(A)$. Consequently $\gamma(A) \geqslant \Omega$, and comparing this formula with Th. 61 we obtain at once

(9) $$\gamma(A) = \Omega.$$

The formulas (5) and (9) form the conclusion of the theorem.

The theorem and the next corollary immediately following were obtained jointly by Lindenbaum and the author.

THEOREM 63. (*a*) *If* $\gamma(A) \leqslant \omega$, *then* $\mathfrak{g}(A) \leqslant \aleph_0$;
(*b*) *if* $\gamma(A) \geqslant \omega$, *then* $\mathfrak{g}(A) \geqslant \aleph_0$.

Proof. (*a*) If A is not a subset of S, then by Th. 59 *a* $\mathfrak{g}(A) = 0$, and so $\mathfrak{g}(A) \leqslant \aleph_0$. Hence, we restrict ourselves to the case where

(1) $$A \subseteq S.$$

We put

(2) $$F(X) = Cn(A+X) \quad \textit{for every set } X \subseteq S.$$

By an easy argument (which is completely analogous with the derivation of (3) in the previous proof) we infer from Th. 6 with the help of (1) and (2) that Axs. 1–4 as well as all their consequences remain valid if 'Cn' is everywhere replaced by 'F'. In particular, by Def. 2, Th. 16 can be transformed in the following way:

(3) *for every set $X \subseteq S$ there is a corresponding sequence of sentences x_ν of type $\pi \leqslant \omega$ which satisfies the formulas*

$$(\alpha)\ x_\nu \in S - F\Big(\underset{x_\mu}{E}[\mu < \nu]\Big) \quad \textit{for } \nu < \pi$$

and

$$(\beta)\ F\Big(\underset{x_\nu}{E}[\nu < \pi]\Big) = F(X).$$

By Th. 57 the type π of every sequence of sentences which satisfies the formula (α) of the condition (3) is less than $\gamma(A)$, thus according to the hypothesis $< \omega$. Further by Th. 7 it follows from (2) that for every set $X \in \mathfrak{Q}(A).\mathfrak{S}$ the formula $F(X) = X$ holds. In view of this we obtain from (3)

(4) *for every set $X \in \mathfrak{Q}(A).\mathfrak{S}$ there is a corresponding sequence of sentences $x_\nu \in S$ of type $\pi < \omega$ which satisfies the formula*

$$F\Big(\underset{x_\nu}{E}[\nu < \pi]\Big) = X.$$

According to Ax. 1 the set S is at most denumerable; hence, as is well known, every set of finite sequences consisting only of elements of S is likewise at most denumerable.

Now (4) shows that the function F maps such a set of sequences onto the class $\mathfrak{Q}(A).\mathfrak{S}$; consequently this class is also at most denumerable. From this by Def. 8a it follows immediately that

$$\mathfrak{g}(A) \leqslant \aleph_0, \quad \text{q.e.d.}$$

(b) By Th. 57 the hypothesis implies that

$$(5) \qquad A \subseteq S,$$

and that, with every number $\pi < \omega$ (thus $< \gamma(A)$), a sequence of sentences $x_\nu^{(\pi)}$ of type π can be correlated so that

$$(6) \qquad x_\nu^{(\pi)} \in S - Cn\Big(A + \underset{x_\mu^{(\pi)}}{E}[\mu < \nu]\Big) \quad \textit{for} \quad \nu < \pi < \omega.$$

(The axiom of choice is not used in this step, for, as we have already mentioned, the set of all finite sequences of sentences is at most denumerable.)

We now put

$$(7) \qquad X_\nu^{(\pi)} = Cn\Big(A + \underset{x_\mu^{(\pi)}}{E}[\mu < \nu]\Big) \quad \textit{for} \quad \nu < \pi < \omega,$$

and let

$$(8) \qquad \mathfrak{K} = \underset{X_\nu^{(\pi)}}{E}[\nu < \pi < \omega].$$

By the use of Th. 9a it easily results from (4)–(8) that

$$(9) \qquad \mathfrak{K} \subseteq \mathfrak{Q}(A).\mathfrak{S}.$$

From (8) it follows at once that $\overline{\overline{\mathfrak{K}}} \leqslant \aleph_0$. But on the other hand the class $\mathfrak{K}$ is infinite. For by Ax. 2 we infer without difficulty from (5)–(7) that $x_\mu^{(\pi)} \in X_\nu^{(\pi)} - X_\mu^{(\pi)}$ and therefore $X_\mu^{(\pi)} \neq X_\nu^{(\pi)}$ for $\mu < \nu < \pi < \omega$; hence by (8) we obtain $\overline{\overline{\mathfrak{K}}} \geqslant \bar{\pi}$ for every $\pi < \omega$. Accordingly

$$(10) \qquad \overline{\overline{\mathfrak{K}}} = \aleph_0.$$

In view of Def. 8a the formulas (9) and (10) give the required inequality

$$\mathfrak{g}(A) \geqslant \aleph_0.$$

Corollary 64. (a) *If $\gamma(A) = \omega$, then $\mathfrak{g}(A) = \aleph_0$;* [Th. 63$a$, b]

(b) *if $\mathfrak{g}(A) < \aleph_0$, then $\gamma(A) < \omega$;* [Th. 63b]

(c) *if $\mathfrak{g}(A) > \aleph_0$, then $\gamma(A) > \omega$.* [Th. 63a]

Corollary 65. *For every set $A \subseteq S$ we have*

$$\textit{either} \quad 1 \leqslant \mathfrak{g}(A) \leqslant \aleph_0 \quad \textit{or} \quad \aleph_0 \leqslant \mathfrak{g}(A) \leqslant 2^{\aleph_0}.$$

[Ths. 59a, 61, 63a, b]

From this corollary Th. 30 is at once derivable as a special case (for $A = 0$).

Theorem 66. *If $\gamma(A) \neq \Omega$, then $\overline{\gamma(A)} \leqslant \mathfrak{g}(A)$.*

Proof. In the case $\gamma(A) = 0$ the conclusion is obvious. If $0 < \gamma(A) < \omega$, then the class $\mathfrak{Q}(A).\mathfrak{S}$, by virtue of Def. 8$a$, contains at least $\overline{\gamma(A)-1}$ consistent systems and moreover, by Th. 47, one contradictory system, namely S. Thus we have $\overline{\overline{\mathfrak{Q}(A).\mathfrak{S}}} \geqslant \overline{\gamma(A)-1}+1 = \overline{\gamma(A)}$, which by Def. 8$b$ gives the formula $\overline{\gamma(A)} \leqslant \mathfrak{g}(A)$. Finally, if we have $\omega \leqslant \gamma(A) < \Omega$, then $\overline{\gamma(A)} = \aleph_0$ and, by Th. 63b, $\mathfrak{g}(A) \geqslant \aleph_0$; from this again the conclusion follows: $\overline{\gamma(A)} \leqslant \mathfrak{g}(A)$. Since by Th. 61 and the hypothesis $\gamma(A) < \Omega$, all possible cases have been dealt with and so the theorem is proved.

It remains undecided whether this theorem can be extended to the case $\gamma(A) = \Omega$.

The following theorem is of a more special nature.

THEOREM 67. *Let $S \in \mathfrak{A}$; then, if π is a limit number, we have $\gamma(A) \neq \pi+1$ for every set A.*

Proof. Let us suppose, in contradiction with the conclusion, that for a certain set A

(1) $$\gamma(A) = \pi+1.$$

Hence, by Def. 8b, for the number π (as well as for every number $\xi < \pi+1$) there exists a sequence of sets X_ν of type π which satisfies the formulas:

(2) $$X_\nu \in \mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W} \quad \textit{for} \quad \nu < \pi,$$

(3) $$X_\mu \subseteq X_\nu \quad \textit{and} \quad X_\mu \neq X_\nu \quad \textit{for} \quad \mu < \nu < \pi.$$

Let us put

(4) $$X_\pi = \sum_{\nu<\pi} X_\nu.$$

Let $\mathfrak{K} = \underset{X_\nu}{E}[\nu < \pi]$. Since $\pi \neq 0$, we also have $\mathfrak{K} \neq 0$; from this by (3) we conclude that the class $\mathfrak{K}$ satisfies condition (α) of Cor. 5. Thus, by (2), $\mathfrak{K} \subseteq \mathfrak{S}.\mathfrak{W}$ holds. Consequently we can apply Ths. 12 and 49 to this class; thus by the formula $S \in \mathfrak{A}$ assumed in the hypothesis we obtain $\sum_{X \in \mathfrak{K}} X \in \mathfrak{S}.\mathfrak{W}$. But since, by (2) and (4), $A \subseteq X_\pi = \sum_{X\in\mathfrak{K}} X$, we have $X_\pi \in \mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W}$, and accordingly (2) can be generalized in the following way:

(5) $$X_\nu \in \mathfrak{Q}(A).\mathfrak{S}.\mathfrak{W} \quad \textit{for} \quad \nu < \pi+1.$$

Since by hypothesis π is a limit number, (4) provides an analogous generalization of (3):

(6) $$X_\mu \subseteq X_\nu \quad \textit{and} \quad X_\mu \neq X_\nu \quad \textit{for} \quad \mu < \nu < \pi+1.$$

In consequence of Def. 8b the existence of a sequence of

sets (of type $\pi+1$) satisfying the formulas (5) and (6) contradicts the assumption (1). We must have therefore

$$\gamma(A) \neq \pi+1, \quad \text{q.e.d.}$$

The results obtained here which concern the notions $\mathfrak{g}(A)$ and $\gamma(A)$ are rather fragmentary. Only by restricting ourselves to those deductive disciplines which presuppose the sentential calculus are we able to obtain more complete results. In particular we can then show that, for every set A of sentences, $\mathfrak{g}(A)$ is either finite or equals $2^{\aleph_0}$, and similarly $\gamma(A)$ is either finite or equals Ω.

In general, when developing the metamathematics of deductive disciplines which presuppose the sentential calculus, we chiefly concentrate upon the same notions which have been discussed in the present paper. Since, however, the development is based upon logically stronger (though more special) assumptions, various results obtained here can be supplemented and improved.

POSTSCRIPT

More recently, and rather unexpectedly, it has turned out that some results of this article can be considerably improved without strengthening the axiomatic basis of our discussion (and thus without restricting ourselves to those deductive disciplines which presuppose the sentential calculus). In particular, Cor. 65 can now be replaced by the following result:

For every set $A \subseteq S$ we have
either $1 \leqslant \mathfrak{g}(A) \leqslant \aleph_0$ or else $\mathfrak{g}(A) = 2^{\aleph_0}$.

Hence, taking the empty set for A we obtain an analagous improvement of Th. 30:

Either $1 \leqslant \overline{\overline{\mathfrak{S}}} \leqslant \aleph_0$ or $\overline{\overline{\mathfrak{S}}} = 2^{\aleph_0}$.

Thus we can say that the cardinality of the class $\mathfrak{S}$ *satisfies the continuum hypothesis*. This proves also to apply to other classes discussed in this article, in fact, to $\mathfrak{A}$, $\mathfrak{W}$, $\mathfrak{U}$, $\mathfrak{A}\mathfrak{q}$ (A), and (trivally) $\mathfrak{Q}(B)$ and $\mathfrak{P}(C)$ for fixed subsets A,B,C of S. The classes obtained from those just mentioned by forming finite (or even denumerable) unions, finite (or denumerable) intersections, and complements with respect to $\mathfrak{P}(S)$ continue to enjoy the property discussed. All of these results have either been announced in the abstracts Burris-Kwatinetz (7b) and (7c), or can be obtained by means of the methods outlined in the second abstract.

connected with the investigations of Gödel. As is well known, Gödel has developed a method which makes it possible, in every theory which includes the arithmetic of natural numbers as a part, to construct sentences which can be neither proved nor disproved in this theory. But he has also pointed out that the undecidable sentences constructed by this method become decidable if the theory under investigation is enriched by the addition of variables of higher type. The proof that the sentences involved actually in this way become decidable again rests on the definition of truth. Similarly—as I have shown by means of the methods used in developing semantics—for any given deductive theory it is possible to indicate concepts which cannot be defined in this theory, although in their content they belong to the theory, and become definable in it if the theory is enriched by the introduction of higher types. Summarizing, we can say that the establishment of scientific semantics, and in particular the definition of truth, enables us to match the negative results in the field of metamathematics with corresponding positive ones, and in that way to fill to some extent the gaps which have been revealed in the deductive method and in the very structure of deductive science.[1]

[1] More detailed information about many of the problems discussed in this article can be found in VIII. Attention should also be called to my later paper, Tarski, A. (82). While the first part of that paper is close in its content to the present article, the second part contains polemical remarks regarding various objections which have been raised against my investigations in the field of semantics. It also includes some observations about the applicability of semantics to empirical sciences and their methodology.

XVI

ON THE CONCEPT OF LOGICAL CONSEQUENCE†

THE concept of *logical consequence* is one of those whose introduction into the field of strict formal investigation was not a matter of arbitrary decision on the part of this or that investigator; in defining this concept, efforts were made to adhere to the common usage of the language of everyday life. But these efforts have been confronted with the difficulties which usually present themselves in such cases. With respect to the clarity of its content the common concept of consequence is in no way superior to other concepts of everyday language. Its extension is not sharply bounded and its usage fluctuates. Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which are connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or less degree.

Even until recently many logicians believed that they had succeeded, by means of a relatively meagre stock of concepts, in grasping almost exactly the content of the common concept of consequence, or rather in defining a new concept which coincided in extent with the common one. Such a belief could easily arise amidst the new achievements of the methodology of deductive science. Thanks to the progress of mathematical logic we have learnt, during the course of recent decades, how to present mathematical disciplines in the shape of formalized deductive theories. In these theories, as is well known, the

† BIBLIOGRAPHICAL NOTE. This is a summary of an address given at the International Congress of Scientific Philosophy in Paris, 1935. The article first appeared in print in Polish under the title 'O pojciu wynikania logicznego' in *Przegląd Filozoficzny*, vol. 39 (1936), pp. 58–68, and then in German under the title 'Über den Begriff der logischen Folgerung', *Actes du Congrès International de Philosophie Scientifique*, vol. 7 (Actualités Scientifiques et Industrielles, vol. 394), Paris, 1936, pp. 1–11.

proof of every theorem reduces to single or repeated application of some simple rules of inference—such as the rules of substitution and detachment. These rules tell us what transformations of a purely structural kind (i.e. transformations in which only the external structure of sentences is involved) are to be performed upon the axioms or theorems already proved in the theory, in order that the sentences obtained as a result of such transformations may themselves be regarded as proved. Logicians thought that these few rules of inference exhausted the content of the concept of consequence. Whenever a sentence follows from others, it can be obtained from them—so it was thought—in more or less complicated ways by means of the transformations prescribed by the rules. In order to defend this view against sceptics who doubted whether the concept of consequence when formalized in this way really coincided in extent with the common one, the logicians were able to bring forward a weighty argument: the fact that they had actually succeeded in reproducing in the shape of formalized proofs all the exact reasonings which had ever been carried out in mathematics.

Nevertheless we know today that the scepticism was quite justified and that the view sketched above cannot be maintained. Some years ago I gave a quite elementary example of a theory which shows the following peculiarity: among its theorems there occur such sentences as:

A_0. *0 possesses the given property P*,

A_1. *1 possesses the given property P*,

and, in general, all particular sentences of the form

A_n. *n possesses the given property P*,

where 'n' stands for any symbol which denotes a natural number in a given (e.g. decimal) number system. On the other hand the universal sentence:

A. *Every natural number possesses the given property P*,

cannot be proved on the basis of the theory in question by means of the normal rules of inference.[1] This fact seems to me to speak

[1] For a detailed description of a theory with this peculiarity see IX; for the discussion of the closely related rule of infinite induction see VIII, pp. 258 ff.

for itself. It shows that the formalized concept of consequence, as it is generally used by mathematical logicians, by no means coincides with the common concept. Yet intuitively it seems certain that the universal sentence A follows in the usual sense from the totality of particular sentences $A_0, A_1, \ldots, A_n, \ldots$. Provided all these sentences are true, the sentence A must also be true.

In connexion with situations of the kind just described it has proved to be possible to formulate new rules of inference which do not differ from the old ones in their logical structure, are intuitively equally infallible, i.e. always lead from true sentences to true sentences, but cannot be reduced to the old rules. An example of such a rule is the so-called rule of infinite induction according to which the sentence A can be regarded as proved provided all the sentences $A_0, A_1, \ldots, A_n, \ldots$ have been proved (the symbols 'A_0', 'A_1', etc., being used in the same sense as previously). But this rule, on account of its infinitistic nature, is in essential respects different from the old rules. It can only be applied in the construction of a theory if we have first succeeded in proving infinitely many sentences of this theory—a state of affairs which is never realized in practice. But this defect can easily be overcome by means of a certain modification of the new rule. For this purpose we consider the sentence B which asserts that all the sentences $A_0, A_1, \ldots, A_n, \ldots$ are *provable* on the basis of the rules of inference hitherto used (not that they have actually been proved). We then set up the following rule: if the sentence B is proved, then the corresponding sentence A can be accepted as proved. But here it might still be objected that the sentence B is not at all a sentence of the theory under construction, but belongs to the so-called metatheory (i.e. the theory *of* the theory discussed) and that in consequence a practical application of the rule in question will always require a transition from the theory to the metatheory. In order to avoid this objection we shall restrict consideration to those deductive theories in which the arithmetic of natural numbers can be developed, and observe that in every such theory all the concepts and sentences of the corresponding metatheory can be interpreted (since a one-one correspondence can be established between expressions of a language and natural numbers).[1] We can replace in the rule discussed the sentence B by the sentence B', which is the arithmetical interpretation of B. In this way we reach a rule which does not deviate essentially from the rules of inference, either in the conditions of its applicability or in the nature of the concepts involved in its formulation or, finally, in its intuitive infallibility (although it is considerably more complicated).

Now it is possible to state other rules of like nature, and even as many of them as we please. Actually it suffices in fact to notice that the rule last formulated is essentially dependent upon the extension of the concept 'sentence provable on the basis of the rules hitherto used'. But when we adopt this rule we thereby widen the extension of this concept. Then, for the widened extension we can set up a new, analogous rule, and so on *ad infinitum*. It would be interesting to investigate whether there are any objective reasons for assigning a special position to the rules ordinarily used.

The conjecture now suggests itself that we can finally succeed in grasping the full intuitive content of the concept of consequence by the method sketched above, i.e. by supplementing the rules of inference used in the construction of deductive theories. By making use of the results of K. Gödel[2] we can show that this conjecture is untenable. In every deductive theory (apart from certain theories of a particularly elementary nature), however much we supplement the ordinary rules of inference by new purely structural rules, it is possible to construct sentences which follow, in the usual sense, from the theorems of this theory, but which nevertheless cannot be proved in this theory on the basis of the accepted rules of

[1] For the concept of metatheory and the problem of the interpretation of a metatheory in the corresponding theory see article VIII, pp. 167 ff., 184, and 247 ff.

[2] Cf. Gödel, K. (22), especially pp. 190 f.

inference.[1] In order to obtain the proper concept of consequence, which is close in essentials to the common concept, we must resort to quite different methods and apply quite different conceptual apparatus in defining it. It is perhaps not superfluous to point out in advance that—in comparison with the new—the old concept of consequence as commonly used by mathematical logicians in no way loses its importance. This concept will probably always have a decisive significance for the practical construction of deductive theories, as an instrument which allows us to prove or disprove particular sentences of these theories. It seems, however, that in considerations of a general theoretical nature the proper concept of consequence must be placed in the foreground.[2]

The first attempt to formulate a precise definition of the proper concept of consequence was that of R. Carnap.[3] But this

[1] In order to anticipate possible objections the range of application of the result just formulated should be determined more exactly and the logical nature of the rules of inference exhibited more clearly; in particular it should be exactly explained what is meant by the structural character of these rules.

[2] An opposition between the two concepts in question is clearly pointed out in article IX, pp. 293 ff. Nevertheless, in contrast to my present standpoint, I have there expressed myself in a decidedly negative manner about the possibility of setting up an exact formal definition for the proper concept of consequence. My position at that time is explained by the fact that, when I was writing the article mentioned, I wished to avoid any means of construction which went beyond the theory of logical types in any of its classical forms; but it can be shown that it is impossible to define the proper concept of consequence adequately whilst using exclusively the means admissible in the classical theory of types; unless we should thus limit our considerations solely to formalized languages of an elementary and fragmentary character (to be exact, to the so-called languages of finite order; cf. article VIII, especially pp. 268 ff.). In his extremely interesting book, Carnap, R. (10), the term (*logical*) *derivation* or *derivability* is applied to the old concept of consequence as commonly used in the construction of deductive theories, in order to distinguish it from the concept of *consequence* as the proper concept. The opposition between the two concepts is extended by Carnap to the most diverse derived concepts ('f-*concepts*' and 'a-*concepts*', cf. pp. 88 ff., and 124 ff.); he also emphasizes—to my mind correctly—the importance of the proper concept of consequence and the concepts derived from it, for general theoretical discussions (cf. e.g. p. 128).

[3] Cf. Carnap, R. (10), pp. 88 f., and Carnap, R. (11) especially p. 181. In the first of these works there is yet another definition of consequence which is adapted to a formalized language of an elementary character. This definition is not considered here because it cannot be applied to languages of a more complicated logical structure. Carnap attempts to define the concept of logical consequence not only for special languages, but also within the framework of what he calls '*general syntax*'. We shall have more to say about this on p. 416, note 1.

attempt is connected rather closely with the particular properties of the formalized language which was chosen as the subject of investigation. The definition proposed by Carnap can be formulated as follows:

The sentence X follows logically *from the sentences of the class K if and only if the class consisting of all the sentences of K and of the negation of X is contradictory.*

The decisive element of the above definition obviously is the concept 'contradictory'. Carnap's definition of this concept is too complicated and special to be reproduced here without long and troublesome explanations.[1]

I should like to sketch here a general method which, it seems to me, enables us to construct an adequate definition of the concept of consequence for a comprehensive class of formalized languages. I emphasize, however, that the proposed treatment of the concept of consequence makes no very high claim to complete originality. The ideas involved in this treatment will certainly seem to be something well known, or even something of his own, to many a logician who has given close attention to the concept of consequence and has tried to characterize it more precisely. Nevertheless it seems to me that only the methods which have been developed in recent years for the establishment of scientific semantics, and the concepts defined with their aid, allow us to present these ideas in an exact form.[2]

Certain considerations of an intuitive nature will form our starting-point. Consider any class K of sentences and a sentence X which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class K consists only of true sentences and the sentence X is false. Moreover, since we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the

[1] See footnote 3 on p. 413.

[2] The methods and concepts of semantics and especially the concepts of truth and satisfaction are discussed in detail in article VIII; see also article XV.

objects spoken about in the sentence X or the sentences of the class K. The consequence relation cannot be destroyed by replacing the designations of the objects referred to in these sentences by the designations of any other objects. The two circumstances just indicated, which seem to be very characteristic and essential for the proper concept of consequence, may be jointly expressed in the following statement:

(F) *If, in the sentences of the class K and in the sentence X, the constants—apart from purely logical constants—are replaced by any other constants (like signs being everywhere replaced by like signs), and if we denote the class of sentences thus obtained from K by 'K′', and the sentence obtained from X by 'X′', then the sentence X′ must be true provided only that all sentences of the class K′ are true.*

[For the sake of simplifying the discussion certain incidental complications are disregarded, both here and in what follows. They are connected partly with the theory of logical types, and partly with the necessity of eliminating any defined signs which may possibly occur in the sentences concerned, i.e. of replacing them by primitive signs.]

In the statement (F) we have obtained a necessary condition for the sentence X to be a consequence of the class K. The question now arises whether this condition is also sufficient. If this question were to be answered in the affirmative, the problem of formulating an adequate definition of the concept of consequence would be solved affirmatively. The only difficulty would be connected with the term 'true' which occurs in the condition (F). But this term can be exactly and adequately defined in semantics.[1]

Unfortunately the situation is not so favourable. It may, and it does, happen—it is not difficult to show this by considering special formalized languages—that the sentence X does not follow in the ordinary sense from the sentences of the class K although the condition (F) is satisfied. This condition may in fact be satisfied only because the language with which we are

[1] See footnote 2 on p. 414.

dealing does not possess a sufficient stock of extra-logical constants. The condition (F) could be regarded as sufficient for the sentence X to follow from the class K only if the designations of all possible objects occurred in the language in question. This assumption, however, is fictitious and can never be realized.[1] We must therefore look for some means of expressing the intentions of the condition (F) which will be completely independent of that fictitious assumption.

Such a means is provided by semantics. Among the fundamental concepts of semantics we have the concept of the *satisfaction of a sentential function* by single objects or by a sequence of objects. It would be superfluous to give here a precise explanation of the content of this concept. The intuitive meaning of such phrases as: *John and Peter satisfy the condition 'X and Y are brothers'*, or *the triple of numbers* 2, 3, *and* 5 *satisfies the equation* '$x+y = z$', can give rise to no doubts. The concept of satisfaction—like other semantical concepts—must always be relativized to some particular language. The details of its precise definition depend on the structure of this language. Nevertheless, a general method can be developed which enables us to construct such definitions for a comprehensive class of formalized languages. Unfortunately, for technical reasons, it would be impossible to sketch this method here even in its general outlines.[2]

One of the concepts which can be defined in terms of the concept of satisfaction is the concept of *model*. Let us assume that in the language we are considering certain variables correspond to every extra-logical constant, and in such a way that every sentence becomes a sentential function if the constants in it are replaced by the corresponding variables. Let L be any class of sentences. We replace all extra-logical constants which

[1] These last remarks constitute a criticism of some earlier attempts to define the concept of formal consequence. They concern, in particular, Carnap's definitions of logical consequence and a series of derivative concepts (L-consequences and L-concepts, cf. Carnap, R. (10), pp. 137 ff.). These definitions, in so far as they are set up on the basis of 'general syntax', seem to me to be materially inadequate, just because the defined concepts depend essentially, in their extension, on the richness of the language investigated.

[2] See footnote 2 on p. 414.

occur in the sentences belonging to L by corresponding variables, like constants being replaced by like variables, and unlike by unlike. In this way we obtain a class L' of sentential functions. An arbitrary sequence of objects which satisfies every sentential function of the class L' will be called a *model* or *realization of the class L of sentences* (in just this sense one usually speaks of models of an axiom system of a deductive theory). If, in particular, the class L consists of a single sentence X, we shall also refer to a model of the class L as a *model of the sentence X*.

In terms of these concepts we can define the concept of logical consequence as follows:

The sentence X follows logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X*.†

It seems to me that everyone who understands the content of the above definition must admit that it agrees quite well with common usage. This becomes still clearer from its various consequences. In particular, it can be proved, on the basis of this definition, that every consequence of true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences. In brief, it can be shown that the condition (F) formulated above is necessary if the sentence X is to follow from the sentences of the class K. On the other hand, this condition is in general not sufficient, since the concept of consequence here defined (in agreement with the standpoint we have taken) is independent of the richness of the language being investigated.

Finally, it is not difficult to reconcile the proposed definition with that of Carnap. For we can agree to call a class of sentences

† After the original of this paper had appeared in print, H. Scholz in his article 'Die Wissenschaftslehre Bolzanos, Eine Jahrhundert-Betrachtung', *Abhandlungen der Fries'schen Schule*, new series, vol. 6, pp. 399–472 (see in particular p. 472, footnote 58) pointed out a far-reaching analogy between this definition of consequence and the one suggested by B. Bolzano about a hundred years earlier.

contradictory if it possesses no model. Analogously, a class of sentences can be called *analytical* if every sequence of objects is a model of it. Both of these concepts can be related not only to classes of sentences but also to single sentences. Let us assume further that, in the language with which we are dealing, for every sentence X there exists a negation of this sentence, i.e. a sentence Y which has as a model those and only those sequences of objects which are not models of the sentence X (this assumption is rather essential for Carnap's construction). On the basis of all these conventions and assumptions it is easy to prove the *equivalence of the two definitions*. We can also show —just as for Carnap—that those and only those sentences are analytical which follow from every class of sentences (in particular from the empty class), and those and only those are contradictory from which every sentence follows.[1]

I am not at all of the opinion that in the result of the above discussion the problem of a materially adequate definition of the concept of consequence has been completely solved. On the contrary, I still see several open questions, only one of which—perhaps the most important—I shall point out here.

Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand, no objective grounds are known to me which permit us to draw a sharp

[1] Cf. Carnap, R. (10), pp. 135 ff., especially Ths. 52.7 and 52.8; Carnap, R. (11), p. 182, Ths. 10 and 11. Incidentally I should like to remark that the definition of the concept of consequence here proposed does not exceed the limits of syntax in Carnap's conception (cf. Carnap, R. (10), pp. 6 ff.). Admittedly the general concept of satisfaction (or of model) does not belong to syntax; but we use only a special case of this concept—the satisfaction of sentential functions which contain no extra-logical constants, and this special case can be characterized using only general logical and specific syntactical concepts. Between the general concept of satisfaction and the special case of this concept used here approximately the same relation holds as between the semantical concept of true sentence and the syntactical concept of analytical sentence.

boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage. In the extreme case we could regard all terms of the language as logical. The concept of *formal* consequence would then coincide with that of *material* consequence. The sentence X would in this case follow from the class K of sentences if either X were true or at least one sentence of the class K were false.[1]

In order to see the importance of this problem for certain general philosophical views it suffices to note that the division of terms into logical and extra-logical also plays an essential part in clarifying the concept 'analytical'. But according to many logicians this last concept is to be regarded as an exact formal correlate of the concept of *tautology* (i.e. of a statement

[1] It will perhaps be instructive to juxtapose the three concepts: 'derivability' (cf. p. 413, note 2), 'formal consequence', and 'material consequence', for the special case when the class K, from which the given sentence X follows, consists of only a finite number of sentences: $Y_1, Y_2, \ldots, Y_n$. Let us denote by the symbol 'Z' the conditional sentence (the implication) whose antecedent is the conjunction of the sentences $Y_1, Y_2, \ldots, Y_n$ and whose consequent is the sentence X. The following equivalences can then be established:

the sentence X is (logically) derivable from the sentences of the class K if and only if the sentence Z is logically provable (i.e. derivable from the axioms of logic);

the sentence X follows formally from the sentences of the class K if and only if the sentence Z is analytical;

the sentence X follows materially from the sentences of the class K if and only if the sentence Z is true.

Of the three equivalences only the first can arouse certain objections; cf. article XII, pp. 342–64, especially 346. In connexion with these equivalences cf. also Ajdukiewicz, K. (2), p. 19, and (4), pp. 14 and 42.

In view of the analogy indicated between the several variants of the concept of consequence, the question presents itself whether it would not be useful to introduce, in addition to the special concepts, a general concept of a relative character, and indeed the concept of *consequence with respect to a class L of sentences*. If we make use again of the previous notation (limiting ourselves to the case when the class K is finite), we can define this concept as follows:

the sentence X follows *from the sentences of the class K* with respect to the class L of sentences *if and only if the sentence Z belongs to the class L.*

On the basis of this definition, derivability would coincide with consequence with respect to the class of all logically provable sentences, formal consequences would be consequences with respect to the class of all analytical sentences, and material consequences those with respect to the class of all true sentences.

which 'says nothing about reality'), a concept which to me personally seems rather vague, but which has been of fundamental importance for the philosophical discussions of L. Wittgenstein and the whole Vienna Circle.[1]

Further research will doubtless greatly clarify the problem which interests us. Perhaps it will be possible to find weighty objective arguments which will enable us to justify the traditional boundary between logical and extra-logical expressions. But I also consider it to be quite possible that investigations will bring no positive results in this direction, so that we shall be compelled to regard such concepts as 'logical consequence', 'analytical statement', and 'tautology' as relative concepts which must, on each occasion, be related to a definite, although in greater or less degree arbitrary, division of terms into logical and extra-logical. The fluctuation in the common usage of the concept of consequence would—in part at least—be quite naturally reflected in such a compulsory situation.

[1] Cf. Wittgenstein, L. (91), Carnap, R. (10), pp. 37–40.

XVII

SENTENTIAL CALCULUS AND TOPOLOGY†

In this article I shall point out certain formal connexions between the sentential calculus and topology (as well as some other mathematical theories). I am concerned in the first place with a topological interpretation of two systems of the sentential calculus, namely the ordinary (two-valued) and the intuitionistic (Brouwer–Heyting) system. With every sentence $\mathfrak{A}$ of the sentential calculus we correlate, in one-one fashion, a sentence $\mathfrak{A}_1$ of topology in such a way that $\mathfrak{A}$ is provable in the two-valued calculus if and only if $\mathfrak{A}_1$ holds in every topological space. An analogous correlation is set up for the intuitionistic calculus. The present discussion seems to me to have a certain interest not only from the purely formal point of view; it also throws an interesting light on the content relations between the two systems of the sentential calculus and the intuitions underlying these systems.

In order to avoid possible misunderstandings I should like to emphasize that I have not attempted to adapt the methods of reasoning used in this article to the requirements of intuitionistic logic.[1] For valuable help in completing this work I am indebted to Professor A. Mostowski.

[1] Most results of this article were obtained in the year 1935. The connexion between the intuitionistic calculus and Boolean algebra (or the theory of deductive systems, see § 5) was discovered by me still earlier, namely in 1931. Some remarks to this effect can be found in article XII of the present book and in Tarski, A. (80). Only after completing this paper did I become acquainted with the work, then newly published, of Stone, M. H. (70). In spite of an entirely different view of Brouwerian logic there is certainly some connexion between particular results of the two works, as can easily be seen comparing Stone's Th. 7, p. 22, and my Th. 4.11. In their mathematical content these two theorems are closely related. But this does not at all apply to the two works as wholes. In particular, Th. 4.24, in which I see the kernel of this paper, tends in quite a different direction from Stone's considerations.

† Bibliographical Note. This article is the text of an address given by the author on 30 September 1937 to the Third Polish Mathematical Congress in Warsaw (see *Annales de la Société polonaise de mathématique*, vol. 16 (1937), p. 192). The article first appeared under the title 'Der Aussagenkalkül und die Topologie', in *Fundamenta Mathematicae*, vol. 31 (1938), pp. 103–34.